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Superdiffusion in Nearly Stratified Flows

Marco Avellaneda[1] **and Andrew J. Majda**[2]

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In classical work, Mathèron and the Marsilly showed that superdiffusive scaling of mean-square displacements occurs in transport diffusion for stratified flows with steady simple shear layers and long-range spatial correlations. More recently the authors have calculated a formula for the non-Gaussian large-scale long-time renormalized Green function for these problems. Here the scaling laws and renormalized Green functions for diffusion in "nearly stratified" flows are studied; in such flows the simple shear layer with long-range correlations is perturbed by incompressible flows with short-range correlations. Here it is established that these flows belong to the same universality class as the simple shear layers, with a renormalized Green function with a similar structure but reflecting homogenization by the transverse displacements. The tools in the analysis involve a modification of homogenization theory and also rigorous diagrammatic perturbation theory.

KEY WORDS: Superdiffusion; anomalous transport; random flows; homogenization.

1. INTRODUCTION

The stochastic Langevin equation

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{V}(\mathbf{X}(t)) + (2D)^{1/2}\,\eta(t) \tag{1}$$

where $\mathbf{V}(\mathbf{x})$ is a random, divergence-free velocity field and $\boldsymbol{\eta}(t)$ is a δ-correlated white noise, arises in many situations in mathematical physics, most notably in the description of the motion of tagged particles in flow through porous media. For simplicity, we will assume that the velocity is

[1] Courant Institute, New York, New York 10012.

[2] Program in Computational and Applied Mathematics, Princeton University, Princeton, New Jersey 08540.

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statistically homogeneous and has mean zero. It is well known since the work of Taylor that the presence of a stirred ambient fluid enhances the rate of dispersion of particles and, in this regard, two generic universality classes of random velocities have been identified. These universality classes correspond to either *diffusive* behavior at long time scales or *superdiffusive* behavior. For the first class of velocity statistics, the mean-square displacement

$$\sigma^2(t) = E\{\mathbf{X}^2(t)\}$$

$[E\{\cdot\}$ denotes averaging with respect to the random force $\boldsymbol{\eta}(t)]$ satisfies

$$\lim_{t \to \infty} \frac{\langle \sigma^2(r) \rangle}{t} = 2dD^*$$

where D^* is an effective diffusion coefficient such that $D < D^* < +\infty$, d is the spatial dimensionality, and $\langle \cdot \rangle$ denotes averaging with respect to velocity statistics. For the second class of velocity statistics, the mean-square displacement is *super*linear, i.e.,

$$\langle \sigma^2(t) \rangle \gg t, \qquad t \to \infty$$

These two categories of velocity statistics differ greatly in the statistical properties of the paths $\mathbf{X}(t)$ for large t. For diffusive velocity statistics, it is known that the solution of the Langevin equation (1) is *self-averaging* in the sense that the normalized, *unaveraged* mean-square displacements converge, i.e.,

$$\lim_{t \to \infty} \frac{\sigma^2(t)}{t} = 2dD^* \tag{2}$$

in probability (and thus almost surely a subsequence), so that sample-to-sample velocity fluctuations are irrelevant as $t \to \infty$. More generally, it can be shown that the Green function, or probability density $P(\mathbf{x}, t)$ for the position of a particle which starts at $\mathbf{x} = \mathbf{0}$ at $t = 0$, satisfies

$$\lim_{\delta \downarrow 0} \delta^{-d} P\left(\frac{\mathbf{x}}{\delta}, \frac{t}{\delta^2}\right) = \frac{1}{(4\pi D^* t)^{d/2}} e^{-|\mathbf{x}|^2/4D^*t} \tag{3}$$

in probability, where D^* is the effective diffusivity in (2). An essentially necessary and sufficient condition for incompressible, mean-zero velocities to give rise to diffusive behavior is that the Eulerian two-point correlation function $R_{\alpha\beta}(\mathbf{x}) = \langle V_\alpha(\mathbf{x})\, V_\beta(\mathbf{0}) \rangle$ satisfy the condition

$$\int_0^\infty dt \left[\int_{\mathbf{R}^d} R_{\alpha\alpha}(\mathbf{x}) \frac{e^{-|\mathbf{x}|^2/4Dt}}{(4\pi D t)^{d/2}} d^d\mathbf{x} \right] < +\infty$$

or, equivalently,

$$\int_{\mathbf{R}^d} \frac{d\mu(\mathbf{k})}{|\mathbf{k}|^2} < +\infty \tag{4}$$

where $d\mu(\mathbf{k})$ is the Fourier transform of $R_{\alpha\alpha}(\mathbf{x})$. This result was established in ref. 10. This condition characterizes the universality class for which diffusive behavior holds; it states quantitatively that the ballistic motions arising from long-wavelength components of the velocity field are balanced by the molecular diffusion and have a negligible effect on the long-time motion. For this class of velocities, the Lagrangian particle has a finite mean relaxation time in which it "samples" completely the random field $\mathbf{V}(\mathbf{x})$, after which it reaches the asymptotic Gaussian limit for every realization of $\mathbf{V}(\mathbf{x})$. Condition (4) is also equivalent to the existence of a statistically homogeneous vector potential $\mathbf{A}$ satisfying

$$\nabla \times \mathbf{A}(\mathbf{x}) = \mathbf{V}(\mathbf{x})$$

such that

$$\langle |\mathbf{A}(\mathbf{x})|^2 \rangle = \langle |\mathbf{A}(\mathbf{0})|^2 \rangle < +\infty$$

Superdiffusion occurs when the integral in (4) diverges and thus any vector potential of $\mathbf{V}(\mathbf{x})$ will necessarily have larger and larger fluctuations as $|\mathbf{x}| \to \infty$. The study of computer simulations and a few tractable models shows that sample-to-sample velocity fluctuations are then critical in determining the behavior of $\sigma^2(t)$ and $P(x, t)$ at large times/distances. Moreover, the superlinear exponent of the average mean-square displacement $\langle \sigma^2(t) \rangle$ depends on the rates of infrared divergence of $\mu(d\mathbf{k})/|\mathbf{k}|^2$, or equivalently on the fluctguations in the amplitude of the vector potential $\langle |\mathbf{A}(\mathbf{x})|^2 \rangle$ for large $\mathbf{x}$. Because of this, the universality class of superdiffusive velocity fields is in fact composed of infinitely many "subclasses" according to the infrared behavior of the velocity statistics.

This phenomenon was illustrated in the work of Mathéron and de Marsilly[1] on dispersion of pollutants in groundwaters. These authors considered a class of "stratified" velocity statistics in two space dimensions of the form

$$\mathbf{V}(x, y) = \begin{pmatrix} V_1(y) \\ 0 \end{pmatrix} \tag{5}$$

for which the velocity correlation function $\langle V_1(y)\, V_1(0) \rangle = R(y)$ satisfies

$$\int_0^y (y - s)\, R(s)\, ds \sim |y|^\varepsilon a^{2-\varepsilon}, \qquad y \to \infty \tag{6}$$

where $0<\varepsilon<2$, and a represents a typical length scale. The velocity potential (stream function)

$$\psi(y)=\int_0^y V_1(s)\,ds$$

grows as $y\to\infty$, since

$$\langle|\psi(y)|^2\rangle=\left\langle\left|\int_0^y V_1(y)\,dy\right|^2\right\rangle=2\int_0^y (y-s)\,R(s)\,ds\sim 2\,|y|^{\varepsilon}a^{2-\varepsilon} \tag{6'}$$

Mathéron and de Marsilly showed that the corresponding mean-square displacement $\sigma_x^2(t)=E\{x(t)^2\}$ is superlinear, and that

$$\lim_{t\to\infty}\frac{\langle\sigma_x^2(t)\rangle}{t^{1+\varepsilon/2}}=\frac{C_\varepsilon \bar{V}^2 a^{2-\varepsilon}}{(2D)^{1-\varepsilon/2}} \tag{7}$$

where $\bar{V}^2=\langle|V_1(\mathbf{0})|^2\rangle$ and C_ε is a numerical constant. Here D is a (phenomenological) transverse diffusion coefficient.[1]

The calculation of the asymptotics for the Green functions for such stratified models was done for the first time in ref. 2. We considered a class of random stratified fields of the form (5) with

$$V_1(y)=\bar{V}\int_{-\infty}^{+\infty}|k|^{(1-\varepsilon)/2}\psi_\infty(k)\,e^{iky/a}\,dW(k) \tag{8}$$

where $\psi_\infty(k)$ is an ultraviolet cutoff satisfying $\psi_\infty(k)=1$ and decaying rapidly at infinity. The notation $dW(k)$ in (8) denotes stochastic integration with respect to Gaussian white noise. Our motivation for studying such a model came from the theory of eddy diffusivity for hydrodynamic turbulent transport of passive scalars, and the study comprised a much wider class of statistics, including time-dependent velocities, which are important for turbulence modeling, but are not relevant in this discussion. The assumption of the Gaussianity of $dW(\mathbf{k})$ is not essential and was removed in ref. 3. The characterization of the coarse-grained limit of the Green function for particle displacements in the x direction $P_1(x,t)=\mathrm{Prob}\{X(t)=x\mid X(0)=0\}$ that emerged was that, for each $\varepsilon\in(0,2)$, there exists a time-scaling function $\rho(\delta)=\delta^{1/(1+\varepsilon/2)}$—which corresponds to the exponent in (7)—such that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\left\langle P\left(\frac{x}{\delta},\frac{t}{[\rho(\delta)]^2}\right)\right\rangle=\bar{P}(x,t)$$

where $\bar{P}(x, t)$ is defined via its Fourier transform, $\hat{\bar{P}}(k, t)$,

$$\bar{P}(x, t) = \int_{-\infty}^{+\infty} \hat{\bar{P}}(k, t) e^{ikx}\, dk$$

by

$$\hat{\bar{P}}(k, t) = E\left\{\exp\left[-\frac{\bar{V}^2 t^{1+\varepsilon/2} k^2 a^{2-\varepsilon}}{2^{2-\varepsilon/2} D^{1-\varepsilon/2}} \int_0^1 \int_0^1 F_\varepsilon(\beta(s) - \beta(s'))\, ds\, ds'\right]\right\} \tag{9}$$

The integration in (9) is with respect to Brownian motions $\beta(t)$, $0 < t < 1$, and the function $F_\varepsilon(y)$ appearing in the exponent is given by

$$F_\varepsilon(y) = \int_{-\infty}^{+\infty} |k|^{1-\varepsilon} e^{iky}\, dk$$
$$= \begin{cases} C_\varepsilon \dfrac{\text{sign}(y)}{|y|^{2-\varepsilon}} & \text{for} \quad \varepsilon \neq 1 \\ C_1 \delta(y) & \text{for} \quad \varepsilon = 1 \end{cases}$$

where C_ε, C_1 are numeric al constants. Clearly, $\bar{P}(x, t)$ is non-Gaussian—it is a *mixture* of Gaussians. This result can be regarded as the counterpart of the homogenization theorem (3) for the case of stratified velocities with superdiffusive behavior. Recently, Bouchaud *et al.*[4] and Zumofen *et al.*[13] studied a special case of stratified disorder consisting of an array of infinite parallel "channels" of width a along which the velocity is a random variable that takes the values $\pm \bar{V}$ independently with equal probability, which corresponds to $\varepsilon = 1$, and recovered the characterization (9). These authors put further in evidence the non-Gaussian behavior of $\bar{P}(x, t)$ by computing the higher-order moments and showing that

$$\lim_{t \to \infty} \frac{\langle E(x(t))^{2n} \rangle}{\langle E(x(t))^2 \rangle^n} > \frac{(2n)!}{2^n n!}$$

and by estimating the $x \to \infty$ behavior of $\bar{P}(x, t)$. On the other hand, the fluctuations of the Green function were analyzed by us in ref. 2, where we showed that

$$\lim_{\delta \to 0} \left\langle \left[\frac{1}{\delta} P\left(\frac{x}{\delta}, \frac{t}{\rho(\delta)^2}\right)\right]^n \right\rangle \neq \bar{P}(x, t)^n$$

for all $n > 1$.

A natural question that arises in connection with these stratified models is to determine the universality class of random velocity fields $\mathbf{V}(\mathbf{x})$

which give rise to renormalized Green functions of the form (7). This is an issue of some physical interest if one wants to apply the theory to realistic velocity statistics which are not *exactly* stratified, but yet appear to have long-range correlations only in certain directions. The main contention in this paper is that Green functions analogous to (7) arise generically for a class of random velocity fields which are "nearly stratified," in the sense that they have the structure

$$\mathbf{V}(x, y) = \begin{pmatrix} V_1(y) \\ 0 \end{pmatrix} + \begin{pmatrix} U_1(x, y) \\ U_2(x, y) \end{pmatrix} \tag{10}$$

where $V_1(y)$ satisfies the assumption (6) or (6′) of the stratified models and the vector field $\mathbf{U}(x, y) = (U(x, y),\ U_2(x, y))$ is incompressible, has mean zero, and satisfies the mean-field condition (4) corresponding to normal diffusion. This corresponds to flow in a stratified *heterogeneous* porous medium, the perturbations $\mathbf{U}(x, y)$ varying only over short wavelengths [relative to $V_1(y)$]; see Fig. 1. Intuitively, we expect the solution of the corresponding Langevin equation

$$\begin{aligned} \frac{dx(t)}{dt} &= V_1(y(t)) + U_1(x(t), y(t)) + (2D)^{1/2}\, \eta_1(t) \\ \frac{dy(t)}{dt} &= U_2(x(t), y(t)) + (2D)^{1/2}\, \eta_2(t) \end{aligned} \tag{11}$$

to behave as follows: on a coarse-grained scale, the y component of the path $y(t)$ will approach (statistically) a Brownian motion. Moreover, the contribution of the term $U_1(x(t), y(t))$ to the velocity of the x component should be negligible in the long-time limit. Therefore the system will behave like the Mathéron–de Marsilly simple shear flow and the effective Green function in the x direction should approach the function $\bar{P}(x, t)$ given in (9). However, the y component will "feel" the advection from the component $U_2(x(t), y(t))$ of the velocity, which will enhance its rate of dispersion. Therefore, its coarse-grained limit behavior should be characterized by an effective transverse diffusivity $D^*_{yy} > D$, which is determined by the total velocity field $\mathbf{V}(x, y)$ and thus, in the definition of $\bar{P}(x, t)$, the "bare" diffusivity D should be replaced, self-consistently, by the renormalized transverse diffusivity D^*_{yy}.

A justification of this picture hinges on showing that an effective separation of scales between the diffusive y motion and the superdiffusive x motion exists. Namely, these models should have the property that the $y(t)$ motion "thermalizes" in a time scale which is short with respect to the time scale on which superdiffusion occurs. Moreover, a crucial ingredient in

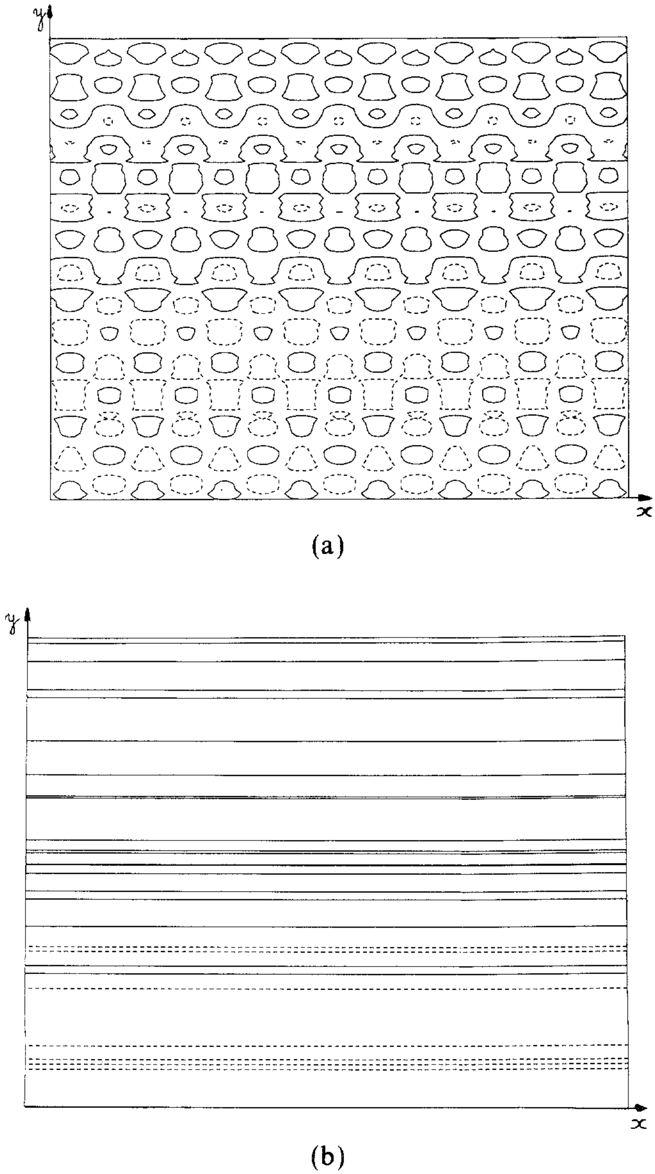


Fig. 1. (a) Level sets of the velocity stream function $\psi(x, y) = W(y) + \sin x \sin y$, where $W(y)$ is a continuous-time random walk. This corresponds to a nearly stratified flow composed of a "purely" stratified $V_1(y)$ component corresponding to $\varepsilon = 1$ and to a perturbation $\mathbf{U}(x, y) = [\sin x \cos y - \cos x \sin y]$. The solid lines represent streamlines corresponding to positive values of $\psi(x, y)$ and the dashed lines represent streamlines corresponding to $\psi < 0$. (b) Level sets of the "unperturbed" stream function $\psi(x, y) = W(y)$.

this self-consistency argument is that sample-to-sample fluctuations in the velocity statistics should not affect the $y(t)$ motion, which thermalizes over (almost all) individual realizations of **V**.

The goal of this paper is to show how all this can be made rigorous. We will show that the longitudinal and transverse Green functions $P_1(x, t) = \Pr\{x(t) = x \mid x(0) = 0\}$ and $P_2(y, t) = \Pr\{y(t) = y \mid y(0) = 0\}$ satisfy, respectively,

$$\lim_{\delta \downarrow 0} \left\langle \delta^{-1} P_1\left(\frac{x}{\delta}, \frac{t}{\rho(\delta)^2}\right)\right\rangle = \bar{P}_1(x, t) \tag{12}$$

and

$$\lim_{\delta \downarrow 0} \delta^{-1} P_2\left(\frac{y}{\delta}, \frac{t}{\delta^2}\right) = (4\pi D_{yy}^*)^{-1/2} \exp\left(-\frac{y^2}{4D_{yy}^* t}\right) \tag{13}$$

in probability, where $\rho(\delta) = \delta^{1/(1+\varepsilon/2)}$, and $\bar{P}_1(x, t)$ denotes the Green function defined in (9) with D replaced by D_{yy}^*, the effective transverse diffusivity. This diffusivity is obtained by solving a "cell problem" (in the sense of homogenization theory),[5,6,10,12] namely, let $\chi(x, y)$ be a suitable solution of

$$D\Delta\chi(x, y) + [V_1(y) + U_1(x, y)] \frac{\partial \chi(x, y)}{\partial x} + U_2(x, y) \frac{\partial \chi(x, y)}{\partial y} = U_2(x, y) \tag{14}$$

Then, D_{yy}^* is given by

$$D_{yy}^* = D\left[1 + \left\langle \left|\frac{\partial \chi(x, y)}{\partial x}\right|^2 + \left|\frac{\partial \chi(x, y)}{\partial y}\right|^2 \right\rangle\right] \tag{15}$$

The mathematical tools used to obtain this result are borrowed in part from homogenization theory. More specifically, we adapt a construction of Papanicolaou and Varadhan[5] for averaging diffusion equations with random coefficients to the present setting of nearly stratified flow. This allows us to show that the motion in the y direction is indeed self-averaging and diffusive, as well as to eliminate the small-wavelength components of the velocity in the x direction and to characterize the effective diffusion coefficient D_{yy}^*.

In Section 2 we introduce a special class of nearly-stratified velocity fields of the form (5) which can be analyzed regorously by the homogeneization method. In these models, the **U** field is assumed to be periodic in the x direction, for technical reasons. The analysis given here generalizes in a straightforward manner to fields that are quasiperiodic in x, using a result due to Kozlov.[6] In Section 3, the diffusive behavior in the y direction is established and in Section 4 we derive the main result, i.e., the characterization of the renormalized Green function $\bar{P}_1(x, t)$. In Section 5, we take a different approach to the renormalization problem, using a Stieltjes integral representation for the matrix elements of the averaged Green function, developed in ref. 7. Here we do away with (quasi) periodicity in the x direction and assume only that $V_1(y)$ satisfies (6) with $0<\varepsilon<2$ and that $\mathbf{U}(x, y)$ satisfies the mean-field condition for normal diffusion (4). This route yields less precise characterizations of the effective Green functions, but gives the correct value of the scaling function $\rho(\delta)=\delta^{1/(1+\varepsilon/2)}$ in a more general setting.

2. STATISTICAL MODEL AND ESTIMATES FOR THE CORRECTORS

In this section we describe a class of velocity statistics corresponding to nearly stratified flows. Following closely Papanicolaou and Varadhan,[5] we use a Hilbert space formalism which is a natural setting for the homogeneization method. Accordingly, let $\{\Omega, \Sigma, P\}$ denote a probability space endowed with a group of measure-preserving transformations $\{\tau_y\}$, $y \in \mathbb{R}$. This group defines an action of the real line $\mathbf{R}$ on the Hilbert space $L^2(\Omega, P)$ of square-integrable functions $\tilde{f}(\omega)$, by means of the operators

$$T_y \tilde{f}(\omega) = \tilde{f}(\tau_y \omega), \qquad y \in \mathbf{R}, \quad \omega \in \Omega$$

We assume that $\{\tau_y\}$ is stochastically continuous, and that (Ω, Σ, P) is separable, so that for any measurable set A of real numbers, $\sup_{y \in A} T_y \tilde{f}(\omega)$ is a measurable random variable (see ref. 5). The action of $\{\tau_y\}_{y \in \mathbf{R}}$ is assumed to be ergodic. This means that if $\tilde{f}(\omega)$ satisfies

$$T_y \tilde{f}(\omega) = \tilde{f}(\omega)$$

almost surely for all $y \in \mathbf{R}$, the $\tilde{f}(\omega) = \text{const}$. Integration with respect to the measure $P(d\omega)$ will be denoted by angular brackets, $\langle \cdot \rangle$.

We define suitable Hilbert space $\mathscr{H}_0$ and $\mathscr{H}_1$ of periodic functions of x with values in $L^2(\Omega, P)$. Accordingly, if p is a given period and

$$\tilde{f}(x, \omega) = \tilde{f}(x+p, \omega), \qquad x \in \mathbf{R}, \qquad \omega \in \Omega$$

we set

$$\hat{\tilde{f}}_m(\omega) = \frac{1}{p} \int_0^p \tilde{f}(x, \omega) e^{-imx}\, dx \tag{16}$$

for $m = 2k\pi/p$, k integer, and we define the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ by

$$\|\tilde{f}\|_0^2 = \sum_m \langle |\hat{\tilde{f}}_m(\omega)|^2 \rangle$$

and

$$\|\tilde{f}\|_1^2 = \sum_m (1+m^2)\langle |\hat{\tilde{f}}_m(\omega)|^2 \rangle$$

The Hilbert spaces $\mathscr{H}_0$ and $\mathscr{H}_1$ are defined as the completions of the spaces of bounded, measurable functions $\tilde{f}(x, \omega)$ under these norms. Clearly,

$$\|\tilde{f}\|_0^2 = \frac{1}{p}\int_0^p \langle |\tilde{f}(x, \omega)|^2 \rangle \, dx$$

and

$$\|\tilde{f}\|_1^2 = \frac{1}{p}\int_0^p \left\langle |\tilde{f}(x, \omega)|^2 + \left|\frac{\partial \tilde{f}}{\partial x}(x, \omega)\right|^2 \right\rangle dx$$

The random velocity $\mathbf{V}$ is introduced next. We consider a function $\tilde{V}_1(\omega)$ in $L^2(\Omega, P)$ and functions $\tilde{U}_1(x, \omega)$ and $\tilde{U}_2(x, \omega)$ in $\mathscr{H}_0$. Random functions $V_1(y)$, $U_1(x, y)$, and $U_2(x, y)$ are then defined by setting

$$V_1(y) = \tilde{V}_1(\tau_y \omega) \tag{17}$$

and

$$U_i(x, y) = \tilde{U}_i(x, \tau_y \omega), \qquad i = 1, 2 \tag{18}$$

These functions are periodic in x and statistically homogeneous in y. [Conversely, to every triple $V_1(y)$, $U_i(x, y)$, $i = 1, 2$, of stationary random processes which are stochastically continuous, ergodic in y, and periodic in x, there exists an abstract probability space (Ω, Σ, P) and a shift operator τ_y, $y \in \mathbf{R}$, such that (17), (18) hold.[5]] We assume that the mean valocity vanishes, so that

$$\langle V_1(y) \rangle = 0$$

and

$$\frac{1}{p}\int_0^p \langle U_i(x, y) \rangle \, dx = 0, \qquad i = 1, 2$$

Finally, we make specific assumptions on the infrared behavior of $V_1(y)$. Namely, we define the potential

$$\psi(y) = \int_0^y V_1(s)\, ds$$

and assume that

$$\langle |\psi(y)|^2 \rangle \sim \bar{V}^2 a^{2-\varepsilon} y^{\varepsilon}, \qquad y \to \infty \tag{19}$$

where $\bar{V}$ is a typical velocity, a is a typical length scale, and $0 < \varepsilon < 2$. Moreover, we will assume that the rescaled process

$$\rho^{\varepsilon/2} \psi\left(\frac{y}{\rho}\right), \qquad 0 \leqslant y < \infty \tag{20}$$

converges in distribution to a Gaussian process with independent increments (see Remark 2 at the end of Section 4). The first hypothesis is precisely the assumption (6′) made by Mathéron and de Marsilly in their calculations of superdiffusive mean-square displacements. The latter one states that the velocity statistics are in the domain of attraction of Gaussian statistics when rescaled consistently with (20). This assumption is satisfied by the velocity fields (8) and a variety of other statistical models.[3] For instance, the well-studied example of a velocity field $V_1(y)$ that is piecewise constant and takes value $\pm \bar{V}$ randomly and independently on layers of width a satisfies these assumptions with $\varepsilon = 1$. The limit as $\rho \to 0$ of the stochastic process (20) defines, loosely speaking, a coarse-grained Gaussian velocity potential $\bar{\psi}(y)$, which is "self-similar." This potential is then necessarily of the form

$$\bar{\psi}(y) = \text{const} \cdot \bar{V} a^{1-\varepsilon/2} \int_0^y \frac{dN(s)}{(y-s)^{\alpha}}$$

with $N(s)$ being a standard Brownian motion and $\alpha = \frac{1}{2}(1-\varepsilon)$, i.e., $\bar{\psi}(y)$ is a fractional Brownian motion. Equivalently,

$$\bar{\psi}(y) = \bar{V} a^{1-\varepsilon/2} \int_{-\infty}^{+\infty} |k|^{-(1+\varepsilon)/2} (e^{iky} - 1)\, dN(k)$$

We note also that, formally, the "coarse-grained" velocity

$$\bar{V}_1(y) = \frac{\partial}{\partial y} \bar{\psi}(y)$$

is given by

$$\bar{V}_1(y) = \bar{V} a^{1-\varepsilon/2} \int_{-\infty}^{+\infty} |k|^{(1-\varepsilon)/2} e^{iky}\, dN(k) \tag{21}$$

and hence is a generalized Gaussian process in the sense of Gel'fand and Vilenkin.[8] Due to the infinite spectral range, which is ultraviolet divergent, $\bar{V}_1(y)$ is not square-integrable for any ε, $0<\varepsilon<2$. For instance, if $\varepsilon=1$, $\bar{V}_1(y)$ is Gaussian white noise, with $\langle \bar{V}_1(y)\, \bar{V}_1(0)\rangle \propto \delta(y)$.

It is convenient to work with the energy spectra of random fields in $\mathscr{H}_0$. Since $\{T_y\}$ is a commutative group of unitary, strongly continuous operators on $L^2(\Omega, \Sigma, P)$,[5] one has the representation

$$T_y=\int_{-\infty}^{+\infty} e^{ik\cdot y}\, d\mathscr{E}_k$$

where $\{\mathscr{E}_k\}$ is a spectral family of projection operators in $L^2(\Omega, P)$. Accordingly, for any field $\tilde{f}(\omega)$ in $L^2(\Omega, P)$,

$$f(y)-T_y\tilde{f}=\int e^{iky}\, d\mathscr{E}_k\, \tilde{f}$$

and thus $d\mathscr{E}_k\tilde{f}$ can be identified with the random spectral measure associated with the stochastic process $f(y)$. Elements of $\mathscr{H}_0$ can also be represented in spectral form, observing that the Fourier coefficient $\hat{\tilde{f}}_m(\omega)$ of $\tilde{f}(x, \omega)$ defined in (17) satisfies

$$T_y\hat{\tilde{f}}_m=\int_{-\infty}^{+\infty} e^{iky}\, d\mathscr{E}_k\, \hat{\tilde{f}}_m$$

Hence, we have

$$f(x, y)=T_y\tilde{f}(x, \omega)=\sum_m \int_{-\infty}^{+\infty} e^{imx+iky}\, d\mathscr{E}_k\, \hat{\tilde{f}}_m$$

and $d\mathscr{E}_k\,\hat{\tilde{f}}_m$ is identified with the spectral measure of $f(x, y)$. We denote by $\mathscr{M}$ the subspace of $\mathscr{H}_0$ composed of fields $f(x, \omega)$ such that

$$\sum_m \int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k\hat{\tilde{f}}_m|^2\rangle}{k^2+m^2}< +\infty \tag{22}$$

The measure $d\langle |\mathscr{E}_k\hat{\tilde{f}}_m|^2\rangle$ can be identified with the Fourier transform of the correlation function

$$R(x, y)=\frac{1}{p}\int_0^p \langle f(x+h, y)\, f(x, 0)\rangle\, dh$$

Hence, inequality (22) is the analogue of condition (4) for the field $f(x, y)$. Accordingly, we will assume throughout this paper that the perturbation $\mathbf{U}(x, y)$ belongs to the space $\mathscr{M}$, i.e.,

$$\sum_m \int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k\hat{\tilde{\mathbf{U}}}_m|^2\rangle}{k^2+m^2}< +\infty \tag{23}$$

For fields in $\mathscr{H}_0$, we have $m = 2k\pi/p$ and hence the condition (23) is equivalent to

$$\int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k \hat{\mathbf{U}}_0|^2 \rangle}{k^2} < \infty$$

Nevertheless, we state it in the general form (23), which is suitable for generalizations to quasiperiodic fields in which Fourier modes may accumulate near $m = 0$.

In the following proposition, we observe that the assumption (19) on the growth of the potential is equivalent to an algebraic rate of divergence of the integral

$$\int \frac{d\langle |\mathscr{E}_k \tilde{V}_1|^2 \rangle}{k^2}$$

This is an elementary consequence of inversion of the Fourier transform.

Proposition 1. The conditions

$$\langle |\psi(y)|^2 \sim y^{\varepsilon}, \qquad y \to \infty$$

and

$$\int_0^y (y-s)\, R(s)\, ds \sim y^{\varepsilon}, \qquad y \to \infty$$

are equivalent to

$$\int_{|k| \geq \sigma} \frac{d\langle |\mathscr{E}_k \tilde{V}_1|^2 \rangle}{|k|^2} \sim \sigma^{-\varepsilon}, \qquad \sigma \to 0 \quad \blacksquare \tag{24}$$

This characterization of the spectrum of $\tilde{V}_1$ will be used in Section 4.

The following lemma establishes the existence of the gradients of the auxiliary functions $\chi(x, y)$ satisfying an equation of the form (14) in the Hilbert space $\mathscr{H}_0$.

Lemma 2. Let $\tilde{F}(x, \omega)$ be an element of $\mathscr{M}$ such that

$$\frac{1}{p} \int_0^p \langle \tilde{F}(x, \omega) \rangle \, dx = 0$$

Then, there exists a field $\mathbf{E}(x, \omega) = [\tilde{E}_1(x, \omega), \tilde{E}_2(x, \omega)]$ in $\mathscr{H}_0$ such that $\mathbf{E}(x, y) = \mathbf{E}(x, \tau_y \omega)$ satisfies the equations

$$\frac{\partial}{\partial x} E_2(x, y) = \frac{\partial}{\partial y} E_1(x, y) \tag{25}$$

and

$$D\frac{\partial}{\partial x}E_1(x, y)+D\frac{\partial}{\partial y}E_2(x, y)$$
$$+[V_1(y)+U_1(x, y)]\,E_1(x, y)+U_2(x, y)\,E_2(x, y)=F(x, y) \quad (26)$$

where $F(x, y)=\tilde{F}(x, \tau_y\omega)$. Moreover, $\tilde{\mathbf{E}}$ has a average zero, i.e.,

$$\frac{1}{p}\int_0^p \langle \mathbf{E}(x, \omega)\rangle\, dx=0$$

and we have the *a priori* estimate

$$\|\tilde{\mathbf{E}}\|_0^2 \leqslant \frac{1}{D^2}\sum_m \int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k \hat{\tilde{F}}_m|^2\rangle}{m^2+k^2} \quad \blacksquare$$

Lemma 3. In addition to the asumptions of Lemma 1, suppose that the fields $(\partial/\partial x)\,U_1(x, y)$, $(\partial/\partial x)\,U_2(x, y)$, and $(\partial/\partial x)\,F(x, y)$ are uniformly bounded, i.e.,

$$\left|\frac{\partial}{\partial x}F(x, y)\right|+\left|\frac{\partial}{\partial x}U_1(x, y)\right|+\left|\frac{\partial}{\partial x}U_2(x, y)\right| \leqslant C_1 < +\infty$$

for some constant C_1. Then $\tilde{\mathbf{E}}$ belongs to $\mathscr{H}_1$ and we have

$$\|\tilde{\mathbf{E}}\|_1^2 \leqslant \frac{1+C_1^2}{D^3}\sum_m \int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k \hat{\tilde{F}}_m|^2\rangle}{k^2+m^2} \quad \blacksquare$$

For a given $\tilde{F}(x, \omega)$ in $\mathscr{M}$, we define the *corrector* $\chi(x, y)$ *corresponding to the function* $F(x, y)=\tilde{F}(x, \tau_y\omega)$ in terms of $\tilde{\mathbf{E}}$ by the formula

$$\chi(x, y)=\int_0^x E_1(s, 0)\, ds+\int_0^y E_2(x, s')\, ds'$$

Using Eq. (25), it follows that

$$\frac{\partial\chi(x, y)}{\partial x}=E_1(x, y)$$

$$\frac{\partial\chi(x, y)}{\partial y}=E_2(x, y)$$

and $\chi(0, 0)=0$, with probability one. By definition, the function $\chi(x, y)$ is periodic in x with period p. Note also that $\chi(x, y)$ is not statistically homogeneous and, in principle, can grow as $y \to \infty$. The next lemma establishes that this growth is sublinear, *uniformly in x*. More precisely, we have the following result.

Lemma 4. Assume the hypotheses of Lemma 3. Then, for each $M > 0$ and $\alpha > 0$,

$$\lim_{\delta \downarrow 0} \Pr \left\{ \sup_{|y| < M} \sup_{x \in \mathbf{R}} \left| \delta \chi \left(x, \frac{y}{\delta} \right) \right| > \alpha \right\} = 0 \quad \blacksquare$$

We will also use the following lemma, which gives an estimate of the variance of $\sup_{|y| \leqslant M} \sup_{x \in \mathbf{R}} |\delta \chi(x, y/\delta)|$.

Lemma 4′. Suppose that F satisfies the assumptions of Lemma 3. Then

$$\left\langle \sup_{|y| \leqslant M} \sup_{x \in \mathbf{R}} \delta^2 \left| \chi \left(x, \frac{y}{\delta} \right) \right|^2 \right\rangle \leqslant \frac{C(1 + M^2)}{D^3} \sum_m \int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k \hat{F}_m|^2 \rangle}{k^2 + m^2}$$

where C is a constant proportional to

$$\sup_{x, y} \left\{ \left| \frac{\partial U_1}{\partial x} \right|^2 + \left| \frac{\partial U_2}{\partial x} \right|^2 + \left| \frac{\partial F}{\partial x} \right|^2 \right\} \quad \blacksquare$$

The proofs of these lemmas are given in the Appendix.

Remark. We assumed here that the "perturbation" of the stratified flow $\mathbf{U}(x, y)$ is a stationary stochastic process in y with values in the space of periodic functions. We point out that the same method can be used to study the cases in which $\mathbf{U}(x, y)$ is periodic in both x and y as well as perturbations which are quasiperiodic in x and/or y. This follows from the general framework of Papanicolaou and Varadhan, to which the reader is referred for details. We point out, however, that the proofs of Lemmas 4 and 4′ given in the Appendix make use of the periodicity in x [see, for instance, Eq. (A.8)]. Some generalizations of these results to quasiperiodic perturbations which satisfy appropriate Diophantine conditions[6] should be straightforward.

3. DIFFUSIVE MOTION IN THE y DIRECTION

The goal of this section is to show that the rescaled stochastic process

$$\delta y \left(\frac{t}{\delta^2} \right), \qquad 0 < t < 1 \tag{27}$$

converges in distribution to Brownian motion $(2D^*_{yy})^{1/2}\,W(t)$ as $\delta \to 0$, where D^*_{yy} is defined in (29). Moreover, we show that (27) is *self-averaging*, in the sense that $W(t)$ is *independent of the velocity statistics*. This is a key point in the subsequent analysis of superdiffusion in the x direction.

The argument that we use to establish diffusive behavior is borrowed from Papanicolaou and Varadham[5] (see also Kozlov[6] and Oelschlager[9]). In addition, we make strong use of the fact that the corrector variance $\langle |\chi(x, y)|^2 \rangle$ grows only in the y direction (Lemma 4).

Let $\chi(x, y)$ denote the corrector corresponding to $\tilde{F}(x, \omega) = \tilde{U}_2(x, \omega)$. Then $\chi(x, y)$ satisfies the equation

$$D\,\Delta\chi(x, y) + [V_1(y) + U_1(x, y)]\,\frac{\partial\chi(x, y)}{\partial x} + U_2(x, y)\,\frac{\partial\chi(x, y)}{\partial y} = U_2(x, y)$$

and $\nabla\chi(x, y) = \mathbf{E}(x, y)$ is such that $\tilde{\mathbf{E}}(x, \omega)$ belongs to the space $\mathscr{H}_1$. Let $(x(t), y(t))$ denote the solution of the stochastic differential equation (11) with initial condition $x(0) = x$, $y(0) = 0$, with x distributed uniformly in the interval $[0, p]$. Applying Itô's formula to $\chi(x(t), y(t))$ we obtain

$$\chi(x(t), y(t)) = \chi(x, 0) + (2D)^{1/2}\int_0^t \frac{\partial\chi(x(s), y(s))}{\partial x}\,d\beta_1(s)$$
$$+ (2D)^{1/2}\int_0^t \frac{\partial\chi}{\partial y}(x(s), y(s))\,d\beta_2(s) + \int_0^t U_2(x(s), y(s))\,ds$$

where $\beta_1(s)$ and $\beta_2(s)$ are independent Brownian motions defined by

$$\beta_i(s) = \int_0^s \eta_i(\tau)\,d\tau, \qquad i = 1, 2$$

Therefore, we have

$$\begin{aligned} y(t) - \chi(x(t), y(t)) &= -\chi(x, 0) + (2D)^{1/2}\,\beta_2(t) \\ &\quad + (2D)^{1/2}\int_0^t \left[\frac{\partial\chi}{\partial x}(x(s), y(s))\,d\beta_1(s) + \frac{\partial\chi}{\partial y}(x(s), y(s))\,d\beta_2(s)\right] \\ &= -\chi(x, 0) + (2D)^{1/2}\,\beta_2(t) + (2D)^{1/2}\int_0^t \mathbf{E}(x(s), y(s))\cdot d\boldsymbol{\beta}(s) \end{aligned}$$

We define the process

$$M(t) = (2D)^{1/2}\left[\beta_2(t) + \int_0^t \mathbf{E}(x(s),\, y(s))\,d\beta(s)\right]$$

which is a continuous-time martingale with quadratic variation

$$Q(t) = 2D \int_0^t \left\{ \left[\frac{\partial \chi}{\partial x}(x(s), y(s)) \right]^2 + \left[1 + \frac{\partial \chi}{\partial y}(x(s), y(s))^2 \right] \right\} ds \tag{28}$$

The process $y(t)$ can be written in the form

$$y(t) = \chi(x(t), y(t)) - \chi(x, 0) + M(t)$$

and hence, rescaling space and time,

$$\delta y\left(\frac{t}{\delta^2}\right) = \delta\chi\left(x\left(\frac{t}{\delta^2}\right), y\left(\frac{t}{\delta^2}\right)\right) - \delta\chi(x, 0) + \delta M\left(\frac{t}{\delta^2}\right)$$

As in usual homogenization arguments,[5,9,12] we show that $\delta M(t/\delta^2)$ converges to a Wiener process, almost surely with respect to the underlying measure induced by the velocity statistics [the probability space (Ω, Σ, P)], and that the remainder

$$\delta\chi\left(x\left(\frac{t}{\delta^2}\right), y\left(\frac{t}{\delta^2}\right)\right) - \delta\chi(x, 0)$$

converges to zero as $\delta \to 0$.

The first part is standard. In fact, $\delta M(t/\delta^2)$ is stochastically equivalent to

$$\delta \tilde{W}\left(Q\left(\frac{t}{\delta^2}\right)\right) \cong \tilde{W}\left[\delta^2 Q\left(\frac{t}{\delta^2}\right)\right]$$

where $\tilde{W}$ is a standard Brownian motion and $Q(t)$ is the quadratic variation in (28). Moreover, since $(x(t), y(t))$ is ergodic with respect to the product measure

$$\frac{dx}{p} \times P(d\omega)$$

by the Birkhoff ergodic theorem, the rescaled quadratic variation

$$\delta^2 Q\left(\frac{t}{\delta^2}\right) = \delta^2 2D \int_0^{t/\delta^2} \left\{ \left[\frac{\partial \chi}{\partial x}(x(s), y(s)) \right]^2 + \left[1 + \frac{\partial \chi}{\partial y}(x(s), y(s))^2 \right] \right\} ds$$

converges with probability one to $2D^*_{yy} t$, where

$$D^*_{yy} = D\left[1 + \frac{1}{p}\int_0^p \langle |\mathbf{E}(x, \omega)|^2 \rangle \, dx\right]$$

This implies that, with probability one with respect to velocity statistics, the paths $\delta M(t/\delta^2)$, $0 \leqslant t \leqslant 1$, converge in distribution to $(2D^*_{yy})^{1/2} W(t)$, where $W(t)$ is standard Brownian motion.[9]

Next, we consider the remainder term,

$$\delta\chi\left(x\left(\frac{t}{\delta^2}\right), y\left(\frac{t}{\delta^2}\right)\right) - \delta\chi(x, 0)$$

It is convenient to introduce the stopping time

$$\theta_N^\delta = \inf\left\{t \leqslant 1, \left|\delta y\left(\frac{t}{\delta^2}\right)\right| = N\right\}$$

Clearly,

$$\sup_{t \leqslant \theta_N} \left|\delta\chi\left(x\left(\frac{t}{\delta^2}\right), y\left(\frac{t}{\delta^2}\right)\right) - \delta\chi(x, 0)\right| \leqslant 2 \sup_{x \in \mathbf{R}} \sup_{|y| \leqslant N} \delta \left|\chi\left(x, \frac{y}{\delta}\right)\right|$$

and hence, applying Lemma 3, we conclude that for all $\alpha > 0$,

$$\begin{aligned}
&\overline{\lim_{\delta \downarrow 0}} \Pr\left\{\sup_{t \leqslant \theta_N} \left|\delta y\left(\frac{t}{\delta^2}\right) - \delta M\left(\frac{t}{\delta^2}\right)\right| > \alpha\right\} \\
&\quad = \overline{\lim_{\delta \downarrow 0}} \Pr\left\{\sup_{t \leqslant \theta_N} \left|\delta\chi\left(x\left(\frac{t}{\delta^2}\right), y\left(\frac{t}{\delta^2}\right)\right) - \delta\chi(x, 0)\right| > \alpha\right\} \\
&\quad \leqslant \lim_{\delta \downarrow 0} \Pr\left\{\sup_{x \in \mathbf{R}} \sup_{|y| \leqslant N} \delta \left|\chi\left(x, \frac{y}{\delta}\right)\right| > \frac{\alpha}{2}\right\} = 0
\end{aligned}$$

This implies that the paths

$$\delta y\left(\frac{t \wedge \theta_N}{\delta^2}\right), \qquad 0 \leqslant t \leqslant 1$$

($\wedge$ = minimum) converge in distribution to $(2D_{yy}^*)^{1/2} W(t \wedge \theta_N)$, for all N. To remove the stopping time θ_N, we define P_N^δ to be the distribution of $\delta y(t \wedge \theta_N/\delta^2)$, $0 \leqslant t \leqslant 1$. Let K denote an arbitrary compact set in the space of continuous paths $C[0, 1]$. Then K is bounded, i.e., $\gamma(t) \in K$ implies $\sup_{t \leqslant 1} |\gamma(t)| \leqslant N$ for some N. Thus,

$$\begin{aligned}
&\lim_{\delta \downarrow 0} \Pr\left\{\left[\delta y\left(\frac{t}{\delta^2}\right), 0 \leqslant t \leqslant 1\right] \in K\right\} \\
&\quad = \lim_{\delta \downarrow 0} \Pr\left\{\left[\delta y\left(\frac{t \wedge \theta_N}{\delta^2}\right), \ 0 \leqslant t \leqslant 1\right] \in K\right\} \\
&\quad = \lim_{\delta \downarrow 0} P_N^\delta(K) \\
&\quad = \Pr\{[(2D_{yy}^*)^{1/2} W(t \wedge \theta_N), 0 \leqslant t \leqslant 1] \in K\} \\
&\quad = \Pr\{[(2D_{yy}^*)^{1/2} W(t), 0 \leqslant t \leqslant 1] \in K\}
\end{aligned}$$

By choosing K large enough, we can make the latter probability arbitrarily close to 1. This implies that the distributions of $[\delta y(t/\delta^2), 0 \leqslant t \leqslant 1]$ are relatively compact (tight), and identifies the limiting distribution as that of $(2D^*_{yy})^{1/2}\, W(t)$, $0 \leqslant t \leqslant 1$. Notice that this argument also shows that

$$\sup_{t \leqslant 1} \left| \delta y\left(\frac{t}{\delta^2}\right) - \delta M\left(\frac{t}{\delta}\right) \right|$$

tends to zero in probability.

For later purposes, we are interested not only in the asymptotic distribution of $\delta y(t/\delta^2)$, $0 \leqslant t \leqslant 1$, but also in the joint limiting distribution of the *pair* of processes

$$\left[\delta y\left(\frac{t}{\delta^2}\right), \tilde{V}_\delta(y)\right], \qquad 0 \leqslant t \leqslant 1, \quad -\infty < y < +\infty$$

where $\langle \tilde{V}_\delta(y)\}$ is a sequence of processes which are measurable with respect to the σ-algebra generated by $V_1(y)$, $y \in \mathbb{R}$. [Intuitively, $\tilde{V}_\delta(y)$ is a function of $V_1(\cdot)$, for each δ and each y.] To study this joint distribution, we observe that it is the same limiting distribution as for the pair

$$\left[\delta M\left(\frac{t}{\delta^2}\right), \tilde{V}_\delta(y)\right], \qquad 0 \leqslant t \leqslant 1, \quad -\infty < y < +\infty$$

since

$$\sup_{t \leqslant 1} \left| \delta y\left(\frac{t}{\delta^2}\right) - \delta M\left(\frac{t}{\delta^2}\right) \right|$$

tends to zero in probability. The following result characterizes the joint distribution.

Proposition 5. (i) The paths $\delta y(t/\delta^2)$, $0 \leqslant t \leqslant 1$, are asymptotically independent of the velocity statistics. More precisely, the joint distribution of $(2D^*_{yy})^{1/2}\, W(t)$, $0 \leqslant t \leqslant 1$, and $\tilde{V}_1(\omega)$ is the product measure

$$\bar{P}^W \otimes P(d\omega)$$

where $\bar{P}^W$ is the distribution of a Brownian motion $(2D^*_{yy})^{1/2}\, W(t)$, $0 \leqslant t \leqslant 1$.

(ii) Let $\{\tilde{V}_\delta(y)\}$ be a sequence of V_1-measurable processes converging in distribution to a process $\bar{V}(y)$ with distribution $\bar{P}^{\bar{V}}$. Then, the joint distribution of $[\delta y(t/\delta^2), \tilde{V}_\delta(y)]$ converges to the product measure

$$\bar{P}^W \otimes \bar{P}^{\bar{V}}$$

In particular, $(2D^*_{yy})^{1/2}\, W(\cdot)$ and $\bar{V}(\cdot)$ are statistically independent. ∎

Proof. Notice that the statement (ii) implies (i), with $\tilde{V}_\delta(y) = V_1(y)$. To prove (ii), we use the fact that if suffices to characterize the limiting distribution of the processes $[\delta M(t/\delta^2),\ \tilde{V}_\delta(y)]$, $0 \leqslant t \leqslant 1$, $y \in \mathbf{R}$. To do this, let $\Phi_1 = \Phi_1(m_1,..., m_r)$ and $\Phi_2 = \Phi_2(v_1,..., v_s)$ denote two bounded, continuous functions depending on a finite number of variables, and set

$$\Phi_1\left[\delta M\left(\frac{\cdot}{\delta^2}\right)\right] = \Phi_1\left[\delta M\left(\frac{t_1}{\delta^2}\right),..., \delta M\left(\frac{t_r}{\delta^2}\right)\right]$$

where $0 \leqslant t_1 < t_2 < \cdots < t_r \leqslant 1$, and

$$\Phi_2[\tilde{V}_\delta(\cdot)] = \Phi_2[\tilde{V}_\delta(y_1),..., \tilde{V}_\delta(y_s)]$$

where $y_1 < y_2 < \cdots < y_s$. From previous arguments in this section, we have

$$\lim_{\delta \downarrow 0} E\left\{\Phi_1\left[\delta M\left(\frac{\cdot}{\delta^2}\right)\right]\right\} = E^{\bar{P}^W}\{\Phi_1[(2D^*_{yy})^{1/2}\, W(\cdot)]\}$$

almost surely with respect to the velocity statistics. Hence,

$$\begin{aligned}\lim_{\delta \downarrow 0} &\left\langle E\left\{\Phi_1\left[\delta M\left(\frac{\cdot}{\delta^2}\right)\right]\right\} \Phi_2[\tilde{V}_\delta(\cdot)]\right\rangle \\ &= \langle E^{\bar{P}^W}\{\Phi_1[(2D^*_{yy})^{1/2}\, W(\cdot)]\}\ \Phi_2[\bar{V}(\cdot)]\rangle \\ &= E^{\bar{P}^W}\{\Phi_1[(2D^*_{yy})^{1/2}\, W(\cdot)\}\langle \Phi_2[\bar{V}(\cdot)]\rangle\end{aligned}$$

where in the last step we used the fact that D^*_{yy} is a constant. This shows that the joint limiting distribution is indeed product measure, as claimed. ∎

4. SUPERDIFFUSION IN THE *x* DIRECTION

The x component of the solution of the Langevin equation (11) with initial condition $x(0) = x$, $y(0) = 0$, where x is distributed uniformly in $[0, p]$, is given by

$$x(t) = x + (2D)^{1/2}\, \beta_1(t) + \int_0^t U_1(x(s), y(s))\, ds + \int_0^t V_1(y(s))\, ds \qquad (29)$$

Introducing the coarse-grained time scale $\rho = \rho(\delta) = \delta^{1/(1+\varepsilon/2)}$, we have

$$\begin{aligned}\delta x\left(\frac{t}{\rho^2}\right) = \delta x + \delta(2D)^{1/2}\, \beta_1\left(\frac{t}{\rho^2}\right) + \delta \int_0^{t/\rho^2} U_1(x(s), y(s))\, ds \\ + \delta \int_0^{t/\rho^2} V_1(y(s))\, ds\end{aligned}$$

Our goal is to show that the asymptotic probability distribution of the random variable $\delta x(t/\rho^2)$ is the function $\bar{P}_1(x, t)$ described in (9), with $D = D^*_{yy}$. Recall that ε is the exponent characterizing the infrared singularity of the energy spectrum of $V_1(y)$ or, equivalently, the growth of the potential $\psi(y) = \int_0^y V_1(s)$, i.e.,

$$\langle |\psi(y)|^2 \rangle \sim \bar{V}^2 a^{2-\varepsilon} y^\varepsilon, \qquad y \to \infty$$

The choice of the superdiffusive scaling function is motivated by the results of refs. 2 and 3, which dealt with "purely stratified" velocities such that $\mathbf{V}(x, y) = (V_1(y), 0)$.

The strategy for evaluating the effective probability distribution consists in, first, eliminating the high-wavenumber components of the velocity which are irrelevant, and, second, evaluating the distribution of the leading term using the self-averaging property of the path $\delta y(t/\delta^2)$, $0 \leqslant t \leqslant 1$. First we show that the quantity

$$\delta x + \delta(2D)^{1/2} \beta_1 \left(\frac{t}{\rho^2}\right) + \delta \int_0^{t/\rho^2} U_1(x(s), y(s))\, ds$$

converges to zero in probability as $\delta \to 0$. In fact, since $U_1(x, y)$ is an element of $\mathcal{M}$, one can construct as associated corrector by setting $F(x, y) = U_1(x, y)$ in Lemma 2. It follows from the arguments of Section 3, that

$$\rho(2D)^{1/2} \beta_1 \left(\frac{t}{\rho^2}\right) + \rho \int_0^{t/\rho^2} U_1(x(s), y(s))\, ds, \qquad 0 \leqslant t \leqslant 1$$

converges in distribution to a Brownian motion $(2D^*)^{1/2} W(t)$, $0 \leqslant t \leqslant 1$ (where D^* is an appropriate diffusion coefficient, not to be confused with D^*_{yy}), so that

$$\delta x + \delta(2D)^{1/2} \beta_1 \left(\frac{t}{\rho^2}\right) + \delta \int_0^{t/\rho^2} U_1(x(s), y(s))\, ds = O\left(\frac{\delta}{\rho}\right) = O(\delta^{\varepsilon/(2+\varepsilon)})$$

as $\delta \to 0$.

We are left with the computation of the asymptotic distribution of

$$\delta \int_0^{t/\rho^2} V_1(y(s))\, ds$$

Recall that

$$V_1(y) = \int_{-\infty}^{+\infty} e^{iky}\, d\mathscr{E}_k\, \tilde{V}_1$$

where the spectral measure $d\langle|\mathscr{E}_k \tilde{V}_1|^2\rangle = d\mu(k)$ satisfies, according to Proposition 1,

$$\int_{|k|\geqslant\kappa} \frac{d\mu(k)}{k^2} \sim \kappa^{-\varepsilon}, \qquad \kappa \to 0$$

It is convenient to define the truncated fields

$$\begin{aligned} V^{>}_{\rho N}(y) &= \int_{|k|\geqslant \rho N} e^{iky}\, d\mathscr{E}_k\, \tilde{V}_1 \\ V^{<}_{\rho N}(y) &= \int_{-\rho N}^{+\rho N} e^{iky}\, d\mathscr{E}_k\, \tilde{V}_1 \end{aligned} \tag{30}$$

where ρ is the time-scaling parameter and N is a large, positive number. Hence, we have

$$\delta \int_0^{t/\rho^2} V_1(y(s))\, ds = \delta \int_0^{t/\rho^2} V^{>}_{\rho N}(y(s))\, ds + \delta \int_0^{t/\rho^2} V^{<}_{\rho N}(y(s))\, ds$$

First, we show that the term involving $V^{>}_{\rho N}$ has a variance bounded by $N^{-\varepsilon}$ as $\delta \to 0$, and thus that the contributions arising from it are negligible. For this, observe that $V^{>}_{\rho N}(y)$ is in $\mathscr{M}$ with

$$\int \frac{d\langle|\mathscr{E}_k \tilde{V}^{<}_{\rho N}|^2\rangle}{k^2} = \int_{|k|\geqslant \rho N} \frac{d\mu(k)}{\kappa^2} \leqslant \text{const} \cdot (\rho N)^{-\varepsilon} \tag{31}$$

Let $\chi^{>}_{\rho N}(x, y)$ be the corrector associated with $F(x, y) = V^{>}_{\rho N}(y)$. Using the arguments of Section 3 (Itô's formula), we have

$$\begin{aligned} &\delta \int_0^{t/\rho^2} V^{>}_{\rho N}(y(s))\, ds \\ &\quad = \delta\chi^{>}_{\rho N}\left(x\left(\frac{t}{\rho^2}\right), y\left(\frac{t}{\rho^2}\right)\right) \\ &\qquad - \delta\chi^{>}_{\rho N}(x, 0) - \delta(2D)^{1/2} \int_0^{t/\rho^2} \nabla\chi^{>}_{\rho N}(x(s), y(s)) \cdot d\boldsymbol{\beta}(s) \end{aligned} \tag{32}$$

The variance of the last stochastic integral is

$$\begin{aligned} &2D\delta^2 \int_0^{t/\rho^2} \frac{1}{p} \langle E\{|\nabla\chi^{>}_{\rho N}(x(s), y(s))|^2\}\rangle\, dx\, ds \\ &\quad = 2D\delta^2 \int_0^{t/\rho^2} \frac{1}{p} \int_0^p \langle|\nabla\tilde{\chi}^{>}_{\rho N}(x, \omega)|^2\rangle\, dx\, ds \end{aligned}$$

$$= 2Dt \frac{\delta^2}{\rho^2} \frac{1}{p} \int_0^p \langle |\nabla \tilde{\chi}_{\rho N}^{>}(x, \omega)|^2 \rangle \, dx$$

$$\leqslant \text{const} \cdot 2Dt \frac{\delta^2}{\rho^2} (\rho N)^{-\varepsilon}$$

$$= \text{const} \cdot (2Dt) N^{-\varepsilon}$$

This estimate uses Lemma 2 and (31). On the other hand, the remainder in (32),

$$r_\delta(t) \equiv \delta \chi_{\rho N}^{>}\left(x\left(\frac{y}{\rho^2}\right), y\left(\frac{t}{\rho^2}\right)\right) - \delta \chi_{\rho N}^{>}(x, 0)$$

converges to zero in probability. To see this, introduce the stopping time $\theta_R = \inf\{t \leqslant 1;\ \rho\, |y(t/\rho^2)| \geqslant R\}$. Then,

$$\Pr\{r_\delta(t) > \alpha\} \leqslant \Pr\{r_\delta(t) > \alpha;\ t \leqslant \theta_R\} + \Pr\{r_\delta(t) > \alpha,\ t > \theta_R\}$$

$$\leqslant \Pr\left\{\sup_{x \in \mathbf{R}} \sup_{|y| < R} \delta \left|\chi_{\rho N}^{>}\left(x, \frac{y}{\rho}\right)\right| > \alpha\right\} + \Pr\{t \geqslant \theta_R\}$$

Given that $\langle |\nabla \tilde{\chi}_{\rho N}^{>}|^2 \rangle \leqslant \text{const} \cdot (\rho N)^{-\varepsilon}$, and that $\delta = \rho^{1+\varepsilon/2}$, the first term converges to zero as $\delta \to 0$, by Lemma 4′. Also, since $\rho y(t/\rho^2)$ is tight in $C[0, 1]$ (cf. Proposition 5),

$$\lim_{R \to \infty} \Pr\{t \geqslant \theta_R\} = 0$$

and hence $r_\delta(t)$ converges to zero in probability. All this shows that

$$\delta \int_0^{t/\rho^2} V_{\rho N}^{>}(y(s))\, ds$$

is irrelevant in the sense that it converges in probability to a random variable with variance $O(N^{-\varepsilon})$.

The last step consists of calculating the asymptotic distribution of the contribution from the infrared modes:

$$\delta \int_0^{t/\rho^2} V_{\rho N}^{<}(y(s))\, ds, \qquad \delta \to 0 \tag{33}$$

To illustrate the basic idea, we first give a proof for the simpler case in which $V(y)$ is a self-similar Gaussian process. After that, a proof based upon the general assumptions on $V(y)$ of Section 2 is carried out.

4.1. Self-Similar Gaussian Fields

We assume here that

$$V_1(y) = \bar{V} a^{1-\varepsilon/2} \int_{-1}^{1} |k|^{(1-\varepsilon)/2} e^{iky} \, dN(k)$$

where $dN(k)$ is a Gaussian white noise, i.e., $N(k)$ is a two-sided Brownian motion, and $\bar{V}$ and a denote, respectively, the typical velocity and length scale. Accordingly, we have

$$V^{<}_{\rho N}(y) = \bar{V} a^{1-\varepsilon/2} \int_{|k| \leqslant \rho N} |k|^{(1-\varepsilon)/2} e^{iky} \, dN(k) \tag{34}$$

The integral in (33) can be rewritten as

$$\frac{\delta}{\rho^2} \int_0^t V^{<}_{\rho N}\left(\rho^{-1} \rho y \left(\frac{s}{\rho^2}\right)\right) ds = \int_0^t V_\delta \left[\rho y \left(\frac{s}{\rho^2}\right)\right] ds \tag{35}$$

with

$$V_\delta(y) = \frac{\delta}{\rho^2} V^{<}_{\rho N}\left(\frac{y}{\rho}\right)$$

Using (34) and the scale invariance of Brownian motion, we find that $V_\delta(y)$ is stochastically equivalent to

$$\tilde{V}(y) = \bar{V} a^{1-\varepsilon/2} \int_{|k| \leqslant N} |k|^{(1-\varepsilon)/2} e^{iky} \, d\tilde{N}(k)$$

where $\tilde{N}(k)$ is again Brownian. Moreover, the stochastic process $\tilde{V}(y)$ is almost surely continuous in y. Therefore, the quantity of interest,

$$\int_0^t \tilde{V}\left(\rho \left(y \left(\frac{s}{\rho^2}\right)\right)\right) ds \tag{36}$$

is a continuous function of the joint process $[\tilde{V}(y), \rho y(s/\rho^2)]$, $y \in \mathbf{R}$, $0 \leqslant s \leqslant t$. From Proposition 5, it follows that the joint distribution of $[\tilde{V}(y), \rho y(s/\rho^2)]$ converges in the space of probability measures on $C[\mathbf{R}] \otimes C[0, 1]$ to the distribution of $(\tilde{V}(y), (2D^*_{yy})^{1/2} W(t))$, $y \in \mathbb{R}$, $0 \leqslant t \leqslant 1$, where $W(t)$ is a Brownian motion which is independent of $\tilde{V}$. Since (36) is a continuous functional, it converges in distribution to the random variable

$$\int_0^t \tilde{V}((2D^*_{yy})^{1/2} W(s)) \, ds$$

In particular, the characteristic function of this random variable is

$$\hat{P}^{(N)}(\xi, t) = \left\langle E\left\{\exp\left\{i\xi \int_0^t \tilde{V}((2D^*_{yy})^{1/2} W(s))\, ds\right\}\right\}\right\rangle$$
$$= E\left\{\exp\left[-\frac{\xi^2}{2}\int_0^t\int_0^t R_N((2D^*_{yy})^{1/2}[W(s) - W(s')])\, ds\, ds'\right]\right\} \tag{37}$$

where $R_N(y)$ is the autocorrelation function of the process $\tilde{V}(y)$, given by

$$R_N(y) = \bar{V}^2 a^{\varepsilon-2} \int_{|k| \leq N} k^{1-\varepsilon} e^{iky}\, dk$$

Introducing the rescaled Brownian motion $\tilde{\beta}(s) = (1/\sqrt{t})\, W(st)$ and rescaling the Wiener integral in (37), we obtain

$$\hat{P}^{(N)}(\xi, t) = E\left\{\exp\left[-\frac{\xi^2 t^{1+\varepsilon/2}\bar{V}^2 a^{2-\varepsilon}}{2^{2-\varepsilon/2} D^{*\,2-\varepsilon/2}_{yy}} F_N(\tilde{\beta})\right]\right\}$$

where

$$F_N(\tilde{\beta}) = \int_{-N}^{+N} |k|^{1-\varepsilon} \left|\int_0^1 e^{ik\tilde{\beta}(s)}\, ds\right|^2 dk \tag{38}$$

This integral is known to converge as $N \to \infty$ on a set of paths $\tilde{\beta}$ of measure one to $\int_0^1 \int_0^1 F_\varepsilon(\tilde{\beta}(s) - \tilde{\beta}(s'))\, ds\, ds'$.[(2)] Hence, the limit

$$\lim_{N\to\infty} \hat{P}^{(N)}(\xi, t) = \hat{\bar{P}}_1(\xi, t)$$

exists; it defines the Fourier transform of $\bar{P}_1(x, t)$. Recalling that

$$\delta x\left(\frac{t}{\rho^2}\right) - \delta \int_0^{t/\rho^2} V^<_{\rho N}(y(s))\, ds$$

converges in probability to a random variable with variance $O(N^{-\varepsilon})$, we conclude that $\delta x(t/\rho^2)$ converges in probability to a random variable with distribution $\bar{P}_1(x, t)$ with Fourier transform

$$\hat{\bar{P}}_1(\xi, t) = E\left\{\exp\left[-\frac{\xi^2 \bar{V}^2 a^{2-\varepsilon}}{2^{2-\varepsilon/2}(D^*_{yy})^{1-\varepsilon/2}} \bar{F}(\tilde{\beta})\right]\right\}$$

where

$$\bar{F}(\beta) = \int_{-\infty}^{+\infty} |k|^{1-\varepsilon} \left|\int_0^1 e^{ik\beta(s)}\, ds\right|^2 dk = \int_0^1 \int_0^1 F_\varepsilon(\tilde{\beta}(s) - \tilde{\beta}(s'))\, ds\, ds'.$$

4.2. Fields Satisfying the Assumptions of Section 2

Here we give an alternative proof of the above result which does not make use of the fact that $V_1(y)$ is Gaussian. We assume, however, that the rescaled field $(\delta/\rho^2)\, V_1(y/\rho)$ is asymptotically Gaussian, in the following sense:

(i) The potential $\psi(y)=\int_0^y V_1(s)\,ds$ satisfies

$$\langle|\psi(y)|^2\rangle \sim \bar{V}^2 a^{2-\varepsilon} y^{\varepsilon}, \qquad y\to\infty \tag{39}$$

(ii) The random process

$$\rho^{\varepsilon/2}\psi\left(\frac{y}{\rho}\right), \qquad 0\leqslant y< +\infty$$

converges in distribution to a Gaussian process $\bar{\psi}(y)$ with independent increments.

As stated in Section 2, $\bar{\psi}(y)$ is then necessarily a fractional Brownian motion, given by the stochastic integral

$$\bar{\psi}(y)=\text{const}\cdot \bar{V}a^{1-\varepsilon/2}\int_0^y \frac{dN(s)}{(y-s)^{(1+\varepsilon)/2}} \tag{39'}$$

where $N(s)$ is a standard Brownian motion. (In ref. 3 we discussed several models of non-Gaussian fields which are consistent with these hypotheses.)

We need to evaluate the asymptotic distribution of

$$\delta\int_0^{t/\rho^2} V_{\rho N}^{<}(y(s))\,ds \tag{40}$$

where $V_{\rho N}^{<}(y)$ is defined in (30). Notice that we can assume withouot loss of generality that the sharp cutoff at $|k|=\rho N$ is replaced by a smooth one, i.e., that

$$V_{\rho N}^{<}(y)=\int_{-\infty}^{+\infty} \hat{f}(\rho Nk)e^{iky}\,d\mathscr{E}_k\,\tilde{V}_1$$

where $\hat{f}(k)$ is a smooth, even function supported in the interval $[-1, 1]$, such that $\hat{f}(0)=1$ and $f(x)=\int e^{ikx}\hat{f}(k)\,dk$ decays rapidly at infinity.

A straightforward calculation shows that

$$\tilde{V}_\delta(y)=\frac{\delta}{\rho^2}\,V_{\rho N}^{<}\left(\frac{y}{\rho}\right)=\rho^{\varepsilon/2-1}\int \rho Mf(\rho Mz)\,V\left(\frac{y}{\rho}-z\right)dz$$

Therefore, making a change of variables in the integral and using integration by parts, we have

$$\tilde{V}_\delta(y) = \rho^{\varepsilon/2} \int M^2 f'(Mz)\, \psi\left(\frac{y-z}{\rho}\right) dz$$

$$= \rho^{\varepsilon/2} \int_{-\infty}^{+\infty} M^2 f'(M(y-z))\psi\left(\frac{z}{\rho}\right) dz \tag{41}$$

Using the estimate (38) and the rapid decay at infinity of $f'(M(y-z))$ it follows easily from the explicit formula (41) that

$$\langle |\tilde{V}_\delta(y)|^2 \rangle \leqslant \text{const}$$

and

$$\langle |\tilde{V}_\delta(y+h) - \tilde{V}_\delta(y)|^2 \rangle \leqslant \text{const} \cdot |h|^2$$

with constants independent of δ. This implies that $\tilde{V}_\delta(y)$ is almost surely continuous, and that the family of processes $\{\tilde{V}_\delta\}$ is tight in $C[0, 1]$. Moreover, from (41), $\tilde{V}_\delta(y)$ converges in distribution to

$$\bar{V}(y) = \int_{-\infty}^{+\infty} M^2 f'(M(y-z))\, \bar{\psi}(z)\, dz$$

The characterization (39′) of $\bar{\psi}(z)$ implies that

$$\bar{V}(y) = \text{const} \cdot \bar{V} a^{1-\varepsilon/2} \int_{-\infty}^{+\infty} \hat{f}(Mk)\, |k|^{(1-\varepsilon)/2} e^{iky}\, dN(k)$$

where $N(k)$ is a Brownian motion. From this point on, the calculation of the asymptotic distribution of (40) is done exactly as in the Gaussian case. This concludes the proof of the main result:

Proposition 6. Assume that $V_1(y)$ is a stationary random process such that $\psi(y) = \int_0^y V_1(s)\, ds$ satisfies

$$\langle |\psi(y)|^2 \rangle \propto \bar{V}^2 a^{2-\varepsilon} |y|^\varepsilon$$

for $0 < \varepsilon < 2$, and that $\rho^{\varepsilon/2} \psi(y/\rho)$ converges in distribution to a Gaussian process with independent increments. Then, the probability density for the x coordinate of a particle evolving according to the Langevin equation (11) satisfies

$$\lim_{\delta \downarrow 0} \left\langle \frac{1}{\delta} P\left(\frac{x}{\delta}, \frac{t}{\rho(\delta)^2}\right) \right\rangle = \bar{P}_1(x, t)$$

with $\rho(\delta)=\delta^{1+\varepsilon/2}$, where the Fourier transform $\hat{\bar{P}}_1(k,t)$ of $\bar{P}_1(x,t)$ is given by

$$\hat{\bar{P}}_1(k,t)=E\left\{\exp\left(-\frac{k^2\bar{V}^2t^{1+\varepsilon/2}a^{2-\varepsilon}}{2^{2-\varepsilon/2}(D^*_{yy})^{1-\varepsilon/2}}\right)\int_0^1\int_0^1 F_\varepsilon(\beta(s)-\beta(s'))\,ds\,ds'\right\}$$

Here D^*_{yy} is the transverse effective diffusivity defined in (29), $F_\varepsilon(y)=\int_{-\infty}^{+\infty}|k|^{1-\varepsilon}e^{iky}\,dk$, and $\beta(s)$, $0\leqslant s\leqslant 1$, is a standard Brownian motion. ∎

Remark 1. For brevity, we have not derived the corresponding limits for the mean-square displacements $\langle\sigma_x^2(t)\rangle=\langle E\{x(t)^2\}\rangle$ and $\sigma_y^2(t)=E\{y(t)^2\}$. These can be obtained by obvious variants of the proofs of the characterizations of the Green functions $\bar{P}_1(x,t)$ and $\bar{P}_2(y,t)$. The corresponding results for the mean-square displacements are

$$\lim_{t\to\infty}\frac{\sigma_y^2(t)}{t}=2D^*_{yy} \tag{42}$$

in probability, and

$$\lim_{t\to\infty}\frac{\langle\sigma_x^2(t)\rangle}{t^{1+\varepsilon/2}}=\frac{C_\varepsilon\bar{V}^2a^{2-\varepsilon}}{(2D^*_{yy})^{1-\varepsilon/2}}, \tag{43}$$

where C_ε is a numerical constant.

Remark 2. The proof of Proposition 6 suggests the intriguing possibility in which $\rho^{\varepsilon/2}\psi(y/\rho)$ does not converge to a Gaussian process as $\rho\to 0$, even though we have $\langle|\psi(y)|^2\rangle\propto|y|^\varepsilon$. These pathological cases, of which quasiperiodic infrared-divergent fields $V_1(y)$ could be examples, will *not* have the same coarse-grained Green functions. In fact, the above arguments show that the distribution of the limit of $\rho^{\varepsilon/2}\psi(y/\rho)$ as $\rho\to\infty$ enters explicitly in the effective Green functions. On the other hand, the anomalous exponent $\rho(\delta)=\delta^{1+\varepsilon/2}$ and the asymptotic relations for the mean-square displacements are expected to hold more generally, independent of the higher-order velocity statistics (see the next section).

5. THE GREEN FUNCTION APPROACH

An alternative approach for studying anomalous diffusion is to considea the average Green function of the corresponding advection-diffusion equation and to exploiot a rigorous resummation procedure for its perturbation expansion to study the limit $\delta\to 0$. This method is quite general; it was developed in ref. 10 for the effective diffusion coefficient under condition (4)oand for Green functions in ref. 7. An advantage of the perturbation expansion method is that it does not require periodicity in x, an assumption that was used in proving Lemmas 4 and 4′ on the sublinear growth of the correctors. The main result that can be obtained in this way is a rigorous justification of the scaling properties of the Langevin equation (diffusive in y, superdiffusive in x) under very general assumptions on V_1 and U. On the other hand, the complete, explicit characterization of the effective Green functions by passing to the limit as $\delta \to 0$ in the perturbation expansion is a cumbersome task. The reflects in part the fact that the effective Green function can depend on higher-order statistics of V_1 if the coarse-grained limit of $\rho^{\varepsilon/2}\psi(y/\rho)$ is non-Gaussian and hence a detailed resummation of all diagrams is required. Fortunately, it is possible to determine the scaling behavior of the system (rigorously) in the case of nearly stratified flows by looking only at a few terms in the expansion. This is what we shall explain here, beginning with a brief review of perturbation theory for the averaged Green function, following ref. 7.

We denote by $P_z(x, y, t)$ the solution of the advection-diffusion equation

$$\frac{\partial P(x, y, t)}{\partial t} + z\mathbf{V}(x, y) \cdot \nabla_{xy} P(x, y, t) = D\, \Delta P(x, y, t)$$

such that

$$P_z(x, y, 0) = \delta(x)\, \delta(y)$$

and by $G_z(x, y, s)$ its Laplace transform,

$$G_z(x, y, s) = \int_0^{+\infty} e^{-st} P_z(x, y, t)\, dt$$

The parameter z in these equations represents a coupling constant which will be set to $z = 1$ in the end. Denote by $\hat{G}_z(k, l, s)$ the Fourier transform of the average Green function $\langle G_z(x, y, s) \rangle$, i.e.,

$$\langle G_z(x, y, s) \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ikx + ily} \hat{G}_z(k, l, s)\, dk\, dl$$

According to the results in ref. 7, $\hat{G}_z(k, l, s)$ is, for each k, l, s, an analytic function of z for all $z = z_1 + iz_2$ in the complement of the imaginary axis $z = iz_2$. More precisely, we have

$$\frac{1}{\hat{G}_z(k, l, s)} = s + Dk^2 + Dl^2 + S_z(k, l, s) \tag{44}$$

where $S_z(k, l, s)$ is representable as a Stieltjes integral

$$S_z(k, l, s) = \int_{-\infty}^{+\infty} \frac{z^2 \nu(k, l, s; d\tau)}{1 + z^2 \tau^2} \tag{45}$$

with respect of a family of positive measures $\nu(k, l, s, d\tau)$. The function $S_z(k, l, s)$ is, in the language of field theory, the sum of all the connected diagrams in the perturbation expansion for $\hat{G}_z(k, l, s)$ in powers of z, and (45) expresses the fact that $\hat{S}_z(k, l, s)$ is summable for arbitrarily large values of the coupling constant z. We are interested in the behavior of $\hat{G}_z(k, l, s)$ in the limit $k \to 0$, $l \to 0$, $s \to 0$. Since the Laplace–Fourier transform of the scaled mean probability density

$$\left\langle \frac{1}{\rho\delta} P\left(\frac{x}{\delta}, \frac{y}{\rho}, \frac{t}{\rho^2}\right)\right\rangle$$

is

$$\rho^2 \hat{G}_z(\delta k, \rho l, \rho^2 s)$$

we will study the limit of this quantity as $\delta \to 0$, $\rho \to 0$ with $\rho = \rho(\delta) = \delta^{1+\varepsilon/2}$. From (44), (45), we have

$$\begin{aligned}
&[\rho^2 \hat{G}_z(\delta k, \rho l, \rho^2 s)]^{-1} \\
&\quad = s + \frac{\delta^2}{\rho^2} Dk^2 + Dl^2 + \frac{1}{\rho^2} \hat{S}_z(\delta k, \rho l, \rho^2 s) \\
&\quad = s + \frac{\delta^2}{\rho^2} Dk^2 + Dl^2 + \frac{1}{\rho^2} \int_{-\infty}^{+\infty} \frac{z^2 \nu(\delta k, \rho l, \rho^2 s; d\tau)}{1 + z^2\tau^2}
\end{aligned} \tag{46}$$

Therefore, the renormalization problem consists in showing that the family of positive measures

$$\nu_\delta(d\tau) = \frac{1}{\rho^2(\delta)} \nu(\delta k, \rho(\delta) l, \rho^2(\delta) s; d\tau), \qquad \delta > 0 \tag{46'}$$

has a nontrivial limit as $\delta \to 0$, in the sense that (possibly along a subsequence of δ) $\nu_\delta(d\tau) \to \bar{\nu}(d\tau)$, where $\bar{\nu}(d\tau) \neq 0$. This will follow if we can show that the zeroth- and second-order moments

$$\int_{-\infty}^{+\infty} \nu_\delta(d\tau) \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \tau^2\, \nu_\delta(d\tau) \tag{47}$$

remain uniformly bounded as $\delta \to 0$, and moreover that $\lim_{\delta \downarrow 0} \int \nu_\delta(d\tau) \neq 0$.[7]

For simplicity in the calculations we shall assume that $V_1(y)$ and $\mathbf{U}(x, y)$ are Gaussian, with $V_1(y)$ as in (8), and that $\mathbf{U}(x, y)$ has absolutely continuous spectral density $\langle |\hat{U}(k, l)|^1 \rangle$ satisfying

$$\iint \frac{\langle |\hat{U}(k, l)|^2 \rangle\, dk\, dl}{k^2 + l^2} < \infty \tag{48}$$

We consider first the quantity $\int_{-\infty}^{+\infty} v_\delta(d\tau)$. From ref. 7, we have

$$\int_{-\infty}^{+\infty} v_\delta(d\tau) = \frac{1}{\rho^2} \int \frac{\delta^2 k^2 \langle |\hat{V}_1(q_2)|^2 \rangle \, dq_2}{\rho^2 s + D|\delta k|^2 + D|\rho l + q_2|^2}$$
$$+ \frac{1}{\rho^2} \iint \frac{\delta^2 k^2 \langle |\hat{U}_1(q_1, q_2)|^2 \rangle \, dq_1 \, dq_2}{\rho^2 s + D|\delta k + q_1|^2 + D|\rho l + q_2|^2}$$
$$+ \frac{1}{\rho^2} \iint \frac{\rho^2 l^2 \langle |\hat{U}_2(q_1, q_2)|^2 \rangle \, dq_1 \, dq_2}{\rho^2 s + D|\delta k + q_1|^2 + D|\rho l + q_2|^2} \tag{49}$$

Clearly,

$$\frac{\delta^2}{\rho^2} \iint \frac{k^2 \langle |\hat{U}_1(q_1, q_2)|^2 \rangle \, dq_1 \, dq_2}{\rho^2 s + D|\delta k + q_1|^2 + D|\rho l + q_2|^2}$$
$$\sim \frac{\delta^2}{\rho^2} k^2 \iint \frac{|\hat{U}_1(q_1, q_2)|^2}{q_1^2 + q_2^2} = O\left(\frac{\delta^2}{\rho^2}\right)$$

and therefore this quantity tends to zero as $\delta \to 0$. Also,

$$\lim_{\delta \downarrow 0} \frac{1}{\rho^2} \iint \frac{\rho^2 l^2 \langle |\hat{U}_2(q_1, q_2)|^2 \rangle \, dq_1 \, dq_2}{\rho^2 s + D|\delta k + q_1|^2 + D|\rho l + q_2|^2}$$
$$= l^2 \iint \frac{\langle |\hat{U}_2(q_1, q_2)|^2 \rangle \, dq_1 \, dq_2}{q_1^2 + q_2^2} < \infty$$

The first integral in (49) is asymptotic to

$$\frac{\delta^2}{\rho^2} \int \frac{k^2 |q_2|^{1-\varepsilon} \, dq_2}{\rho^2 s + D\delta^2 k^2 + D|\rho l + q_2|^2}$$
$$\cong \frac{\delta^2 \rho^{2-\varepsilon}}{\rho^2} \int_{-\infty}^{+\infty} \frac{k^2 |q_2'|^{1-\varepsilon} \, dq_2'}{s^2 + D\delta^2/\rho^2 k^2 + D|l + q_2'|^2}$$
$$\sim \int_{-\infty}^{+\infty} \frac{k^2 |q_2'|^{1-\varepsilon} \, dq_2'}{s^2 + D|l + q_2'|^2}$$

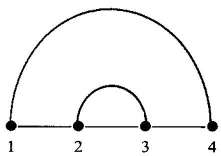
This last quantity is finite for all l (in particular, for $l = 0$), since the integrand decays like $|q_2'|^{-(1+\varepsilon)}$ for $q_2' \gg 1$. The finiteness of $\int_{-\infty}^{+\infty} v_\delta(d\tau)$ implies that the motion in the y direction is diffusive. In fact, setting $k = 0$ in (46), we obtain

$$\lim_{\rho \to 0} [\rho^2 G_z(0, \rho l, \rho^2 s)]^{-1} = s + Dl^2 + \int_{-\infty}^{+\infty} \frac{z^2 \bar{v}(0, l, d\tau)}{1 + z^2 \tau^2}$$

for some measure $\bar{\nu}(0, l, d\tau)$ on $(-\infty, +\infty)$, which shows that $\rho^2 G_z(0, \rho l, \rho^2 s)$ converges to a nontrivial limit.

To analyze the motion in the x direction, we consider the second moments $\int_{-\infty}^{+\infty} \tau^2 \nu_\delta(d\tau)$. Since only macroscopic x displacements are of interest, we can take $l = 0$ in (46′). Using the Gaussianity of $\mathbf{V}(x, y)$, we can write, symbolically,

$$\int \tau^2 \nu_\delta(d\tau) = \text{sum of "bubble" diagrams of the forms}$$



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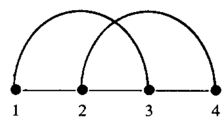
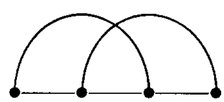


(50)

The graphs in (50) are the ones introduced by Kraichnan,[11] with points denoting velocity modes $\hat{\mathbf{V}}(q_1, q_2)$ and horizontal dashes denoting the free Green function $(s + Dk^2 + Dl^2)^{-1}$. The curved lines denote pairing of modes and averaging. To calculate (50), we observe that diagrams involving only modes $\langle |\hat{\mathbf{U}}(q_1, q_2)|^2 \rangle$ are uniformly bounded, because of the mean-field condition (48)—this was established in ref. 7. Also, notice that the diagrams in (50) which involve only the V_1 field are known to be uniformly bounded, since they arise in the expansion for the Green's function corresponding to the "purely" stratified flow $(V_1(y), 0)$. The study of such diagrams was also done in ref. 7. Therefore, the only diagrams that need to be studied in detail are those that contain *both* V_1 *and* U_1 *or* U_2. The general form of a multiple integral corresponding to the diagram



in the expansion of $(1/\rho^2)\, \hat{S}_z(\delta k, 0, \rho^2 s)$ is

$$\frac{1}{\rho^2} \iiint \frac{[\xi \cdot \mathbf{R}(P) \cdot (\xi + Q)][(\xi + P) \cdot \mathbf{R}(Q) \cdot \xi]\, dP\, dQ}{[\rho^2 s + D|\xi + P|^2][\rho^2 s + D\xi + P + Q|^2][\rho^2 s + D|\xi + Q|^2]} \tag{51}$$

where $\xi = (\delta k, 0)$, $P = (0, p_2)$, $Q = (q_1, q_2)$, and $\mathbf{R}(\cdot)$ represents the two-point correlation tensor. We write a triple integral in (51) because only diagrams containing interactions between $\langle |\hat{V}_1(p_2)|^2 \rangle$ and $\langle |\hat{U}_i(q_1, q_2)|^2 \rangle$ need to be considered. Note also that we take the external wavevector ξ parallel to the x direction. To estimate (51), we observe that

$$\xi \cdot R(P) \cdot (\xi + Q) = \delta k \langle |\hat{V}_1(p_2)|^2 \rangle (\delta k + q_1)$$

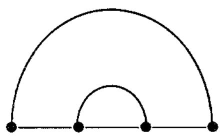
Notice that p_2 and q_1 appear in the numerator of the integrand in (51). Therefore, this integral can be estimated from above by

$$\frac{1}{\rho^2}\iiint \frac{|\xi|^2 \langle |\hat{V}_1(p_2)|^2\rangle \langle |\hat{\mathbf{U}}(q_1, q_2)|^2\rangle \, dp_2 \, dq_1 \, dq_2}{[\rho^2 s + D|\xi + P|^2][\rho^2 s + D|\xi + Q|^2]}$$

$$= \frac{\delta^2 k^2}{\rho^2} \int \frac{\langle |\hat{V}_1(p_2)|^2\rangle \, dp_2}{\rho^2 s + D|\delta k + p_2|^2} \times \iint \frac{|\hat{\mathbf{U}}(q_1, q_2)|^2 \, dq_1 \, dq_2}{\rho^2 s + D|\xi + Q|^2}$$

Making the change of variables $p_2 = \rho p_2'$ in the integral and using $\rho(\delta) = \delta^{1+\varepsilon/2}$ and (48), we conclude that such diagrams are bounded independently of δ, as in the analysis of the second moments.

Similarly, the multiple integrals corresponding to the diagram



are

$$\frac{1}{\rho^2}\iiint \frac{[\xi \cdot R(P) \cdot \xi][(\xi + P) \cdot R(Q) \cdot (\xi + P)] \, dP \, dQ}{[\rho^2 + D|\xi + P|^2]^2 [\rho^2 s + D|\xi + P + Q|^2]} \tag{52}$$

and

$$\frac{1}{\rho^2}\iiint \frac{[(\xi + Q) \cdot R(P) \cdot (\xi + Q)][\xi \cdot R(Q) \cdot \xi] \, dP \, dQ}{[\rho^2 s + D|\xi + Q|^2]^2 [\rho^2 + D|\xi + P + Q|^2]} \tag{53}$$

with the same conventions. These two integrals can be bounded, respectively, by

$$\frac{\delta^2 k^2}{\rho^2}\iiint \frac{\langle |\hat{V}_1(p_2)|^2\rangle \langle |\hat{\mathbf{U}}(q_1, q_2)|^2\rangle \, dp_2 \, dq_1 \, dq_2}{[\rho^2 s + D|\xi + P|^2][\rho^2 s + D|\xi + P + Q|^2]} \tag{54}$$

and

$$\frac{\delta^2 k^2}{\rho^2}\iiint \frac{\langle |\hat{V}_1(p_2)|^2\rangle \langle |\hat{\mathbf{U}}(q_1, q_2)|^2\rangle \, dp_2 \, dq_1 \, dq_2}{[\rho^2 s + D|\xi + Q|^2][\rho^2 s + D|\xi + P + Q|^2]} \tag{55}$$

It now follows, by making the change of variables $p_2 = \rho p_2'$ and using the relation $\tau = \delta^{1+\varepsilon/2}$, that both integrals remain bounded as $\delta \to 0$, converging to

$$k^2 \int \frac{|p_2|^{1-\varepsilon} \, dp_2}{s + Dp_2^2} \times \iint \frac{\langle |\hat{\mathbf{U}}(q_1, q_2)|^2\rangle \, dq_1 \, dq_2}{D(q_1^2 + q_2^2)}$$

Therefore, we have shown that

$$\frac{1}{\rho^2}\int_{-\infty}^{+\infty} \tau^2 \nu(\delta k, 0, \rho^2 s; d\tau) \leqslant c < +\infty$$

where c is a constant independent of δ. The uniform boundedness of the second moment of $(1/\rho^2)\nu(\delta k, 0, \rho^2 s, d\tau)$, together with the result for the zeroth-order moment $\int_{-\infty}^{+\infty} \nu_\delta(d\tau)$, imply that

$$\lim_{\delta \downarrow 0} [\rho^2 G_z(\delta k, 0, \rho^2 s)]^{-1} = s + \int_{-\infty}^{+\infty} \frac{z^2 \bar{\nu}(k, s; d\tau)}{1 + z^2 \tau^2}$$

where $\bar{\nu}(b, s; d\tau)$ is not identically zero.[7] This justifies rigorously the scaling exponent $\rho(\delta) = \delta^{1+\varepsilon/2}$ for the motion in the x direction for this class of nearly stratified flows.

APPENDIX. PROPERTIES OF THE CORRECTORS

The method of homogenization hinges on the construction of the auxiliary functions, or correctors, $\chi(x, y)$. Here we present the proofs of the basic Lemmas 2, 3, 4, and 4′, which were stated in Section 2.

Proof of Lemma 2. According to the classical theory of homogenization for operators with periodic, oscillating coefficients[12] a unique solution of Eqs. (25), (26) exists if $V_1(y)$, $U_1(x, y)$, $U_2(x, y)$, and $F(x, y)$ are periodic in x and y. The idea of the proof of this lemma is to approximate the functions $V_1(y)$, $U_i(x, y)$, and $F(x, y)$ by doubly periodic functions, with period N in the y direction, and to consider the sequence of corresponding correctors $\chi^{(N)}(x, y)$. The final result follows by letting $N \to \infty$, after obtaining suitable estimates. Accordingly, let $V_1^{(N)}(y)$, $U_i^{(N)}(x, y)$, $i = 1, 2$, $F^{(N)}(x, y)$ be periodic with period p in x and period N in y. Moreover, assume that

$$\lim_{N \to \infty} \left\langle \frac{1}{N} \int_0^N |V_1(y) - V_1^{(N)}(y)|^2 \, dy \right\rangle = 0 \tag{A.1}$$

$$\lim_{N \to \infty} \left\langle \frac{1}{pN} \int_0^p \int_0^N |U_i(x, y) - U_i^{(N)}(x, y)|^2 \, dx \, dy \right\rangle = 0 \tag{A.2}$$

$$\lim_{N \to \infty} \left\langle \frac{1}{pN} \int_0^p \int_0^N |F(x, y) - F^{(N)}(x, y)|^2 \, dx \, dy \right\rangle = 0 \tag{A.3}$$

For instance, this can be done by taking a discrete approximation of the spectral measure of the fields, e.g.,

$$V_1^{(N)}(y) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{2\pi i k n/N} \left(\int_{I_n} d\mathscr{E}_k \, \tilde{V}_1 \right)$$

where $I_n = \{k: 2\pi n/N - \pi/N \leqslant k < 2\pi n/N + \pi/N\}$, with similar definitions for the fields $U_i^{(N)}(x, y)$ and $F^{(N)}(x, y)$.

For each realization ω, let $\chi^{(N)}(x, y)$ be the periodic solution of the cell problem

$$D\Delta\chi^{(N)}(x, y) + [V_1^{(N)}(y) + U_1^{(N)}(x, y)] \frac{\partial \chi^{(N)}(x, y)}{\partial x} + U_2^{(N)}(x, y) \frac{\partial \chi^{(N)}(x, y)}{\partial y} = F^{(N)}(x, y) \tag{A.4}$$

with

$$\frac{1}{pN} \int_0^p \int_0^N \chi^{(N)}(x, y) \, dx \, dy = 0$$

Multiplying (A.4) by $\chi^{(N)}(x, y)$ and using integration by parts, one obtains

$$\frac{1}{pN} \int_0^p \int_0^N D |\nabla \chi^{(N)}(x, y)|^2 \, dx \, dy = -\frac{1}{pN} \int_0^p \int_0^N F^{(N)}(x, y) \, \chi^{(N)}(x, y) \, dx \, dy$$

Using Plancherel's identity, we express the RHS in the form

$$\sum_{\substack{m,k \\ m^2 + k^2 \neq 0}} \hat{F}^{(N)}(m, k) \, \overline{\hat{\chi}^{(N)}(m, k)}$$

This quantity is bounded by

$$\left[\sum_{m,k} \frac{|\hat{F}^{(N)}(m, k)|^2}{m^2 + k^2} \right]^{1/2} \left[\sum_{m,k} (m^2 + k^2) |\hat{\chi}^{(N)}(m, k)|^2 \right]^{1/2} \leqslant \left[\sum_{m,k} \frac{|\hat{F}^{(N)}(m, k)|^2}{m^2 + k^2} \right]^{1/2} \left[\frac{1}{pN} \int_0^p \int_0^N |\nabla \chi^{(N)}(x, y)|^2 \, dx \, dy \right]^{1/2}$$

so that

$$\frac{1}{pN}\int_0^p\int_0^N D\,|\nabla\chi^{(N)}(x,y)|^2\,dx\,dy \leqslant \sum_{m,k}\frac{|\hat{F}^{(N)}(x,y)|^2}{D(m^2+k^2)}. \tag{A.5}$$

Setting

$$\mathbf{E}^{(N)}(x,y)=\left[\frac{\partial\chi^{(N)}(x,y)}{\partial x},\frac{\partial\chi^{(N)}(x,y)}{\partial y}\right]$$

we observe that $\mathbf{E}^{(N)}$ satisfies equations analogous to (25) and (26), with coefficients $V_1^{(N)}$, $U_i^{(N)}$, and $F^{(N)}$. Moreover, we have

$$\frac{1}{pN}\int_0^p\int_0^N \langle \mathbf{E}^N(x,y)\rangle\,dx\,dy=0$$

and, averaging (A.5) with respect to the velocity statistics,

$$\frac{1}{pN}\int_0^p\int_0^N \langle|\mathbf{E}^{(N)}(x,y)|^2\rangle \leqslant \frac{1}{D^2}\sum_{m,k}\frac{\langle|\hat{F}^{(N)}(m,k)|^2\rangle}{m^2+k^2}.$$

The right-hand side of this inequality is bounded uniformly, independently of N (because F is in $\mathscr{M}$) and converges to

$$\frac{1}{D^2}\sum_m\int_R\frac{d\langle\mathscr{E}_k\hat{\hat{F}}_m|^2\rangle}{m^2+k^2}$$

as $N\to\infty$. It is easy to conclude from this that the sequence of random fields $\mathbf{E}^N(x,y)$ converge in probability (possibly along a subsequence) to a square-integrable, stationary random field $\mathbf{E}(x,y)$ in the space $\mathscr{H}_0$ (see Section 2) such that $\mathbf{E}(x,y)=\tilde{\mathbf{E}}(x,\tau_y\omega)$ with

$$\|\tilde{\mathbf{E}}\|_0^2\leqslant\frac{1}{D^2}\sum_m\int_R\frac{d\langle|\mathscr{E}_k\hat{\hat{F}}_m|^2\rangle}{m^2+k^2}.$$

Since

$$\frac{\partial}{\partial x}E_2^{(N)}(x,y)=\frac{\partial}{\partial y}E_1^{(N)}(x,y)$$

for all N, we also have

$$\frac{\partial}{\partial x}E_2(x,y)=\frac{\partial}{\partial y}E_1(x,y). \tag{A.6}$$

Finally, due to the strong convergence of the approximating fields [from (A.1)–(A.3)] we deduce from Eq. (A.4) that $\mathbf{E}(x, y)$ saitsfies the equation

$$D\frac{\partial E_1(x,y)}{\partial x}+D\frac{\partial E_2(x,y)}{\partial y}+[V_1(y)+U_1(x,y)]\,E_1(x,y)$$
$$+U_2(x,y)\,E_2(x,y)=F(x,y). \tag{A.7}$$

This concludes the proof of Lemma 2.

Proof of Lemma 3. Here, we exploit the regularity of the coefficients (in x) to show that $(\partial/\partial x)\,\mathbf{E}(x, y)$ is square-integrable, i.e., that $\mathbf{E}(x, y)$ is in the space $\mathscr{H}_1$. To see this, we differentiate Eq. (A.7) with respect to x. Setting $(\partial/\partial x)\mathbf{E}=\mathbf{H}$, we have

$$D\frac{\partial H_1}{\partial x}+D\frac{\partial H_2}{\partial y}+(V_1+U_1)H_1+U_2H_2=\frac{\partial F}{\partial x}-\frac{\partial U_1}{\partial x}E_1-\frac{\partial U_2}{\partial x}E_2$$

We multiply this equation by $E_1(x, y)$, integrate with respect to x, and average. Using (A.6) and the incompressibility of $\mathbf{V}$, we obtain

$$D\left\langle\frac{1}{p}\int_0^p|\mathbf{H}(x,y)|^2\,dx\right\rangle=-\left\langle\frac{1}{p}\int_0^p\frac{\partial F(x,y)}{\partial x}E_1(x,y)\,dx\right\rangle$$
$$+\left\langle\frac{1}{p}\int_0^p\frac{\partial U(x)}{\partial x}E_1(x,y)^2\,dx\right\rangle$$
$$+\left\langle\frac{1}{p}\int_0^p\frac{\partial U_2}{\partial x}(x,y)\,E_2(x,y)\,E_1(x,y)\,dx\right\rangle$$

From the estimate on $\|\mathbf{E}\|_0$ obtained in Lemma 2 and the boundedness of $\partial F/\partial x$, $\partial U_1/\partial x$, and $\partial U_2/\partial x$ we conclude that

$$\left\langle\frac{1}{p}\int_0^p|\mathbf{H}(x,y)|^2\,dx\right\rangle\leqslant\frac{\text{const}}{D}C_1\left\langle\frac{1}{p}\int_0^p|\mathbf{E}(x,y)|^2\,dx\right\rangle$$

where

$$C_1=\sup_{x,y}\left[\left|\frac{\partial F(x,y)}{\partial x}\right|+\left|\frac{\partial U_1(x,y)}{\partial x}\right|+\frac{\partial U_2(x,y)}{\partial x}\right]$$

The conclusion of Lemma 3 is immediate. ∎

Proof of Lemma 4. The goal is to establish the sublinear growth of the function $\chi(x, y)$ *uniformly in* x. More precisely, we shall show that for all $\alpha>0$, $M>0$,

$$\lim_{\delta\downarrow 0}\Pr\left\{\sup_{|y|\leqslant M}\ \sup_x\delta\left|\chi\left(x,\frac{y}{\delta}\right)\right|>\alpha\right\}=0$$

For this, we observe first that since χ is periodic in x,

$$\sup_x \left| \chi(x, y) - \frac{1}{p} \int_0^p \chi(x', y)\, dx' \right|^2 \leqslant \text{const} \cdot \frac{1}{p} \int_0^p \left| \frac{\partial \chi}{\partial x}(x', y) \right|^2 dx' \tag{A.8}$$

Therefore,

$$\sup_x |\chi(x, y)|^2 \leqslant \text{const} \cdot \left[\left| \frac{1}{p} \int_0^p \chi(x, y)\, dx \right|^2 + \frac{1}{p} \int_0^p \left| \frac{\partial \chi}{\partial x}(x, y) \right|^2 dx \right].$$

Setting

$$A(y) = \frac{1}{p} \int_0^p \chi(x, y)\, dx$$

and

$$B(y) = \left\{ \frac{1}{p} \int_0^p \left| \frac{\partial \chi}{\partial x}(x, y) \right|^2 dx \right\}^{1/2},$$

we claim that $\sup_{|y| \leqslant M} \delta |A(y/\delta)|$ and $\sup_{|y| \leqslant M} \delta B(y/\delta)$ both converge to zero in probability as $\delta \to 0$.

The analysis of $\sup_{|y| \leqslant M} \delta A(y/\delta)$ can be done as in the paper of Papanicolaou and Varadhan.[5] We include here a sketch of the proof and refer to ref. 5 for complete details. By definition

$$A(y) = \int_0^y \hat{\tilde{E}}_{2,0}(\tau_s \omega)\, ds$$

where $\hat{\tilde{E}}_{2,0}(\cdot)$ is the Fourier coefficient of order $m = 0$ of the function $\tilde{E}_2(x, \omega)$. Rescaling, we have

$$\delta A\left(\frac{y}{\delta}\right) = \delta \int_0^{y/\delta} \hat{\tilde{E}}_{2,0}(\tau_s \omega)\, ds$$

Since $\langle \hat{\tilde{E}}_{2,0}(\omega) \rangle = 0$ because E_2 has mean zero, we can apply the L^2-ergodic theorem to conclude that

$$\lim_{\delta \downarrow 0} \left\langle \delta^2 \left| A\left(\frac{y}{\delta}\right) \right|^2 \right\rangle = 0 \tag{A.9}$$

for each $y \in \mathbf{R}$. To estimate the supremum of $\delta A(y/\delta)$, $|y| < M$, we partition the interval $[-M, +M]$ into small subintervals and use (A.9) together with equicontinuity of $\delta A(y/\delta)$. Specifically, set

$$y_n = \frac{n}{N}, \qquad -NM \leqslant n \leqslant +NM \tag{A.10}$$

where N is a larger integer, and write

$$\Pr\left\{\sup_{|y|\leqslant M}\left|\delta A\left(\frac{y}{\delta}\right)\right|>\alpha\right\}$$

$$\leqslant\sum_n \Pr\left\{\sup_{|y-y_n|\leqslant 1/N}\delta\left|A\left(\frac{y}{\delta}\right)\right|>\alpha\right\}$$

$$\leqslant\sum_n \Pr\left\{\sup_{|y-y_n|\leqslant 1/N}\left|\delta A\left(\frac{y}{\delta}\right)-\delta A\left(\frac{y_n}{\delta}\right)\right|>\frac{\alpha}{2}\right\}$$

$$+\sum_n \Pr\left\{\delta\left|A\left(\frac{y_n}{\delta}\right)\right|>\frac{\alpha}{2}\right\}$$

From (A.9), the second term converges to zero as $\delta\to 0$. Concerning the first term, we note that

$$\sup_{|y-y_n|\leqslant 1/N}\left|\delta A\left(\frac{y}{\delta}\right)-\delta A\left(\frac{y_n}{\delta}\right)\right|$$

$$=\sup_{|y-y_n|\leqslant 1/N}\delta\left|\int_{y_n/\delta}^{y/\delta}(T_s\hat{\hat{E}}_{2,0})\,ds\right|$$

$$\leqslant\delta\int_{(y_n-1/N)/\delta}^{(y_n+1/N)/\delta}|T_s\hat{\hat{E}}_{2,0}|\,ds$$

Therefore, using Chebyshev's inequality, we obtain

$$\Pr\left\{\sup_{|y-y_n|\leqslant 1/N}\left|\delta A\left(\frac{y}{\delta}\right)-\delta A\left(\frac{y_n}{\delta}\right)\right|>\frac{\alpha}{2}\right\}$$

$$\leqslant\frac{4}{\alpha^2}\left\langle\left[\sup_{|y-y_n|\leqslant 1/N}\left|\delta A\left(\frac{y}{\delta}\right)-\delta A\left(\frac{y_n}{\delta}\right)\right|\right]^2\right\rangle$$

$$\leqslant\frac{4\delta^2}{\alpha^2}\left\langle\left(\int_{(y_n-1/N)/\delta}^{(y_n+1/N)/\delta}|T_s\hat{\hat{E}}_{2,0}|\,ds\right)^2\right\rangle$$

$$\leqslant\frac{4\delta^2}{\alpha^2}\cdot\frac{4}{N^2\delta^2}\langle|\hat{\hat{E}}_{2,0}|^2\rangle$$

$$=\frac{16}{\alpha^2N^2}\langle|\hat{\hat{E}}_{2,0}|^2\rangle$$

where we used Jensen's inequality and the fact that T_s is an isometry. Summing over the contributions of each subinterval $|y-y_n|\leqslant 1/N$, we obtain

$$\sum_n \Pr\left\{\sup_{|y-y_n|\leqslant 1/N}\left|\delta A\left(\frac{y}{\delta}\right)-\delta A\left(\frac{y_n}{\delta}\right)\right|>\frac{\alpha}{2}\right\}\leqslant\frac{32M}{\alpha^2N}\langle|\hat{\hat{E}}_{2,0}|^2\rangle$$

Since N is arbitrary, this implies that $\sup_{|y| \leqslant M} \delta |A(y/\delta)|$ converges to zero in probability.

It remains to estimate the random variable $\sup_{|y| \leqslant M} \delta B(y/\delta)$. This is done in a similar way: first observe that

$$\delta^2 \left\langle B^2\left(\frac{y}{\delta}\right)\right\rangle = \delta^2 \langle |\tilde{E}_1|^2 \rangle$$

by stationarity, so $\delta B(y/\delta)$ converges pointwise to zero. To estimate the supremum for $|y| \leqslant M$, we partition the interval $[-M, M]$ into subintervals of width $O(1/N)$ and use equicontinuity in L^2. Accordingly, if y_n denotes the partition in (A.10), we have

$$\begin{aligned}\sup_{|y| \leqslant 1/M} \delta^2 B\left(\frac{y}{\delta}\right)^2 &\leqslant \max_n \sup_{|y-y_n| \leqslant 1/N} \delta^2 B\left(\frac{y}{\delta}\right)^2 \\ &\leqslant \sum_n \sup_{|y-y_n| \leqslant 1/N} \delta^2 B\left(\frac{y}{\delta}\right)^2 \\ &\leqslant \sum_n \sup_{|y-y_n| \leqslant 1/N} \delta^2 \left| B\left(\frac{y}{\delta}\right) - B\left(\frac{y_n}{\delta}\right)\right|^2 + \sum_n \delta^2 B\left(\frac{y_n}{\delta}\right)^2\end{aligned}$$

Pointwise convergence of $\delta B(y/\delta)$ implies that the second sum in this estimate converges to zero. To the first sum, we observe that

$$\begin{aligned}\delta^2 \left| B\left(\frac{y}{\delta}\right) - B\left(\frac{y_n}{\delta}\right)\right|^2 &\leqslant \frac{\delta^2}{p} \int_0^p \left| \tilde{E}_1\left(x, \frac{y}{\delta}\right) - \tilde{E}_1\left(x, \frac{y_n}{\delta}\right)\right|^2 dx \\ &= \frac{\delta^2}{p} \int_0^p \left| \int_{y_n/\delta}^{y/\delta} \frac{\partial E_1}{\partial y}(x, s)\, ds\right|^2 dx \\ &= \frac{\delta^2}{p} \int_0^p \left| \int_{y_n/\delta}^{y/\delta} \frac{\partial E_2}{\partial x}(x, s)\, ds\right|^2 dx\end{aligned}$$

Therefore,

$$\left\langle \sup_{|y-y_n| \leqslant 1/N} \delta^2 \left| B\left(\frac{y}{\delta}\right) - B\left(\frac{y_n}{\delta}\right)\right|^2 \right\rangle \leqslant \frac{4}{N^2} \|\tilde{E}_2\|_1^2$$

According to Lemma 3, $\|\tilde{E}_2\|_1^2$ is finite. Finally, summing over the contributions of all subintervals, we obtain

$$\sum_n \left\langle \sup_{|y-y_n| \leqslant 1/N} \delta^2 \left| B\left(\frac{y}{\delta}\right) - B\left(\frac{y_n}{\delta}\right)\right|^2 \right\rangle \leqslant \frac{8M}{N} \|\tilde{E}_2\|_1^2$$

which is negligible, since N is arbitrarily small.

This concludes the proof of Lemma 4. ∎

Remark. The estimates obtained in the proof of Lemma 4 also yield an estimate of the variance of $\sup_{|y| \leqslant M} \sup_{x \in \mathbf{R}} \delta\chi(x, y/\delta)$. In fact, taking a trivial partition consisting of one interval and applying the above arguments, one concludes that

$$\left\langle \sup_{|y| \leqslant M} \delta^2 \left| A\left(\frac{y}{\delta}\right)\right|^2 \right\rangle \leqslant \text{const} \cdot (1 + M^2) \|\tilde{\mathbf{E}}\|_0^2$$

and

$$\left\langle \sup_{|y| \leqslant M} \delta^2 B\left(\frac{y}{\delta}\right)^2 \right\rangle \leqslant \text{const} \cdot (1 + M) \|\tilde{\mathbf{E}}\|_1^2$$

Putting together these esimates and using Lemma 3, we conclude that

$$\left\langle \sup_{|y| \leqslant M} \sup_{x \in \mathbf{R}} \delta^2 \left| \chi\left(x, \frac{y}{\delta}\right)\right|^2 \right\rangle \leqslant \text{const} \cdot \frac{1 + M^2}{D^3} \sum_{k,m} \int_{-\infty}^{+\infty} \frac{d\langle |\mathscr{E}_k \hat{\tilde{F}}_m|^2 \rangle}{k^2 + m^2}$$

Hence, the claim of Lemma 4′ in Section 2 is also established.

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REFERENCES

1. G. Mathéron and G. de Marsily, *Water Resources Res.* **16**(5):901–917 (1980).
2. M. Avellaneda and A. J. Majda, *Commun. Math. Phys.* **131**:381–429 (1990).
3. M. Avellaneda and A. J. Majda, *Commun. Math. Phys.* **148** (1992).
4. A. Bouchaud, A. Georges, J. Koplik, S. Redner, and J. Provata, *Phys. Rev. Lett.* **64**:2503 (1990).
5. G. Papanicolaou and S. R. S. Varadhan, in *Random Fields*, J. Fritz, J. J. Lebowitz, and D. Szasz, eds. (North-Holland, Amsterdam, 1982), pp. 835–873.
6. S. M. Kozlov, *Math. USSR Sbornik* **35**:418 (1979).
7. M. Avellaneda and A. J. Majda, in *Composite Media and Homogenization Theory*, G. Dal Maso and G. F. Dell'Antonio, eds. (Birkäuser, Boston, 1991), pp. 13–35.
8. I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions*, Vol. 4 (Academic Press, New York, 1964).
9. K. Oelschlager, *Ann. Prob.* **16**(3):1084–1126 (1988).
10. M. Avellaneda and A. J. Majda, *Commun. Math. Phys.* **138**:339–391 (1991).
11. R. Kraichnan, in *The Padé Approximant in Theoretical Physics*, G. A. Baker and J. L. Gammel, eds. (Academic Press, New York, 1970), p. 129.
12. A. Beusoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis of Periodic Structures* (North-Holland, Amsterdam, 1979).
13. G. Zumofen, J. Klafer, and A. Blumen, preprint; *J. Stat. Phys.* (1992).